# On the decomposition of $n$-partite graphs based on a vertex-removing synchronised graph product 

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#### Abstract

Recently, we have introduced and modified graph-decomposition theorems based on a graph product motivated by applications in the context of synchronising periodic real-time processes. This vertex-removing synchronised product (VRSP) is based on modifications of the well-known Cartesian product and is closely related to the synchronised product due to Wöhrle and Thomas. Here, we introduce a new graph-decomposition theorem based on the VRSP that decomposes an edgelabelled acyclic $n$-partite multigraph where all labels are the same.


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## 1. Introduction

Recently, we have introduced three graph-decomposition theorems in [7], [4] and [2] based on a graph product motivated by applications in the context of synchronising periodic real-time pro-

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cesses, in particular in the field of robotics. More on the background, definitions and applications can be found in two conference contributions [6, 8], four journal papers [2, 4, 7, 9], the thesis of the author [5] and on ArXiv [3]. We repeat some of the definitions in Section2for convenience. In Section 3, we state and prove two lemmas on bipartite and 3-partite graphs which we use to state and prove the decomposition theorem on $n$-partite graphs.

## 2. Terminology and notation

In order to avoid duplication we refer the interested reader to [7] or [3] for background, definitions and more details. Furthermore, we use the textbook of Bondy and Murty [1] for terminology and notation we have not specified here, or in [7] or in [3]. For convenience, we repeat a few definitions that are especially important for the decomposition of an $n$-partite graph.

Let $G$ be an edge-labelled acyclic directed multigraph with a vertex set $V$, an arc set $A$, a set of label pairs $L$ and two mappings. The first mapping $\mu: A \rightarrow V \times V$ is an incidence function that identifies the tail and head of each arc $a \in A$. In particular, $\mu(a)=(u, v)$ means that the arc $a$ is directed from $u \in V$ to $v \in V$, where $\operatorname{tail}(a)=u$ and $h e a d(a)=v$. We also call $u$ and $v$ the ends of $a$. The second mapping $\lambda: A \rightarrow L$ assigns a label pair $\lambda(a)=(\ell(a), t(a))$ to each arc $a \in A$, where $\ell(a)$ is a string representing the (name of an) action and $t(a)$ is the weight of the arc $a$.

If $X \subseteq V(G)$, then the subgraph of $G$ induced by $X$, is the graph on vertex set $X$ containing all the arcs of $G$ which have both their ends in $X$ (together with $L, \mu$ and $\lambda$ restricted to this subset of the arcs).

If $X \subseteq A(G)$, then the subgraph of $G$ arc-induced by $X$ is the graph on arc set $X$ containing all the vertices of $G$ which are an end of an arc in $X$ (together with $L, \mu$ and $\lambda$ restricted to this subset of the arcs).

Let $G_{i}$ and $G_{j}$ be two disjoint graphs. An arc $a \in A\left(G_{i}\right)$ with label pair $\lambda(a)$ is a synchronising $\operatorname{arc}$ with respect to $G_{j}$, if and only if there exists an arc $b \in A\left(G_{j}\right)$ with label pair $\lambda(b)$ such that $\lambda(a)=\lambda(b)$. Furthermore, an arc $a$ with label pair $\lambda(a)$ of $G_{i} \boxtimes G_{j}$ or $G_{i} \boxtimes G_{j}$ is a synchronous arc, whenever there exist a pair of $\operatorname{arcs} a_{i} \in A\left(G_{i}\right)$ and $a_{j} \in A\left(G_{j}\right)$ with $\lambda(a)=\lambda\left(a_{i}\right)=\lambda\left(a_{j}\right)$. Analogously, an arc $a$ with label pair $\lambda(a)$ of $G_{i} \boxtimes G_{j}$ or $G_{i} \boxtimes G_{j}$ is an asynchronous arc, whenever $\lambda(a) \notin L_{i}$ or $\lambda(a) \notin L_{j}$.

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For disjoint nonempty sets $X, Y \subseteq V(G),[X, Y]$ denotes the set of arcs of $G$ with one end in $X$ and one end in $Y$. If the head of the arc $a \in[X, Y]$ is in $Y$, we call $a$ a forward $\operatorname{arc}($ of $[X, Y]$ ); otherwise, we call $a$ a backward arc.

A graph $B$ is called n-partite if there exists a partition of nonempty sets $V_{1}, V_{2}, \ldots, V_{n}$ of $V(B)$ into $n$ partite sets (i.e., $V(B)=V_{1} \cup \ldots \cup V_{n}, V_{i} \cap V_{j}=\varnothing, i \neq j, i, j \in\{1, \ldots, n\}$ ) such that every arc of $B$ has its head vertex and tail vertex in different partite sets. The $n$-partite graph is denoted as $B\left(V_{1}, \ldots, V_{n}\right)$. A bipartite graph $B\left(V_{1}, V_{2}\right)$ is called complete if, for every pair $x \in V_{1}$, $y \in V_{2}$, there is an arc $a$ met $\mu(a)=(x, y)$ or $\mu(a)=(y, x)$ in $B\left(V_{1}, V_{2}\right)$.

Informally, the vertex-removing synchronised product (VRSP) starts from the well-known Cartesian product, and is based on a reduction of the number of arcs and vertices due to the presence of synchronising arcs, i.e., arcs with the same label. This reduction is done in two steps: in the first step synchronising pairs of arcs from $G_{1}$ and $G_{2}$ are replaced by one (diagonal) arc, all other synchronising arcs are removed from the Cartesian product, giving the intermediate product; in the second step, vertices (and the arcs with that vertex as a tail) are removed one by one if they have level $>0$ in the Cartesian product but level $=0$ in what is left of the intermediate product.

## 3. The $\boldsymbol{n}$-partite graph-decomposition theorem.

We assume that the graphs we want to decompose are connected; if not, we can apply our decomposition results to the components separately. Although the decomposition theorems using the VRSP are dealing with edge-labelled graphs where the labels may be different, in this contribution, we consider only acyclic directed graphs where all labels are the same.

We continue with presenting and proving a decomposition lemma of a bipartite graph, given in Lemma 3.1. In this lemma, we are going to decompose a complete bipartite graph $B(X, Y), X=$ $\left\{u_{1}, \ldots, u_{c_{1} \cdot c_{2}}\right\}, Y=\left\{v_{1}, \ldots, v_{c_{3} \cdot c_{4}}\right\}$ where all arcs have the same label. But, different from Lemma 3.1 in [4], we contract both $X$ and $Y$ using disjoint subsets $X_{i}^{\prime}$ of $X$, disjoint subsets $X_{i}^{\prime \prime}$ of $X$, disjoint subsets $Y_{j}^{\prime}$ of $Y$ and disjoint subsets $Y_{j}^{\prime \prime}$ of $Y$ such that $B(X, Y) \cong B(X, Y) /_{i=1}^{c_{1}} X_{i}^{\prime} /_{j=1}^{c_{3}} Y_{j}^{\prime}$ $\triangle B(X, Y) /_{i=1}^{c_{2}} X_{i}^{\prime \prime} /_{j=1}^{c_{4}} Y_{j}^{\prime \prime}$, where $\bigcup_{i=1}^{c_{1}} X_{i}^{\prime}=\bigcup_{i=1}^{c_{2}} X_{i}^{\prime \prime}=X$ and $\bigcup_{j=1}^{c_{3}} Y_{j}^{\prime}=\bigcup_{j=1}^{c_{4}} Y_{j}^{\prime \prime}=Y$. If the cardinality of $X$ is a prime number, hence, $c_{1}=1$ or $c_{2}=1$, then, assuming $c_{2}=1$ and, there-

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fore, $c_{1}=|X|$, the left part of $B(X, Y) /_{i=1}^{c_{1}} X_{i}^{\prime} /_{j=1}^{c_{3}} Y_{j}^{\prime} \nabla B(X, Y) /_{i=1}^{c_{2}} X_{i}^{\prime \prime} /_{j=1}^{c_{4}} Y_{j}^{\prime \prime}$ is contracted such that each vertex $u_{i}$ of $X$ is replaced by the vertex $\tilde{x}_{i}$ and in the right part of $B(X, Y) /_{i=1}^{c_{1}} X_{i}^{\prime} /_{j=1}^{c_{3}} Y_{j}^{\prime}$ $\nabla B(X, Y) /_{i=1}^{c_{2}} X_{i}^{\prime \prime} /_{j=1}^{c_{4}} Y_{j}^{\prime \prime}, \bigcup_{i=1}^{c_{2}} X_{i}^{\prime \prime}=\bigcup_{i=1}^{1} X_{i}^{\prime \prime}=X$ is contracted giving one vertex $\tilde{x}$. We have similar contractions for $Y$ if $|Y|$ is a prime number. Even, if $|X|$ and $|Y|$ are not prime numbers we can set $c_{2}$ and $c_{3}$ to one. This leads to the decomposition $B(X, Y) /_{i=1}^{c_{1}} X_{i}^{\prime} / Y \Delta B(X, Y) / X /_{j=1}^{c_{4}} Y_{j}^{\prime}$ which is equivalent to $B(X, Y) / Y \triangle B(X, Y) / X$. Therefore, Lemma 3.1 is a generalisation of Lemma 3.1 in [4]. Note that for prime numbers for $|X|$ and $|Y|$ the contraction of $X$ to $\tilde{x}$ and $Y$ to $\tilde{y}$ are on opposite sides of the VRSP of $B(X, Y) \int_{i=1}^{c_{1} \cdot c_{2}} X_{i}^{\prime} Y \Delta B(X, Y) / X \int_{j=1}^{c_{3} \cdot c_{4}} Y_{j}^{\prime \prime}$. This is because $B(X, Y) \cong B(X, Y) \int_{i=1}^{c_{1} \cdot c_{2}} X_{i}^{\prime} \int_{j=1}^{c_{3} \cdot c_{4}} Y_{j}^{\prime \prime}$ where $X_{i}^{\prime}=\left\{u_{i}\right\}, Y_{j}^{\prime}=\left\{v_{j}\right\}$, and we do not have a decomposition where the decomposed parts are smaller than $B(X, Y)$.

If the cardinality of $X$ is not a prime number then $X$ is partitioned into $c_{1}$ subsets $X_{i}^{\prime}$ with $\left|X_{i}^{\prime}\right|=c_{2}$ and $X$ is partitioned into $c_{2}$ subsets $X_{i}^{\prime \prime}$ with $\left|X_{i}^{\prime \prime}\right|=c_{1}, Y$ is partitioned into $c_{3}$ subsets $Y_{j}^{\prime}$ with $\left|Y_{j}^{\prime}\right|=c_{4}$ and $Y$ is partitioned into $c_{4}$ subsets $Y_{j}^{\prime \prime}$ with $\left|Y_{j}^{\prime \prime}\right|=c_{3}$. This gives the decomposition $B(X, Y) \cong B(X, Y) /_{i=1}^{c_{1}} X_{i}^{\prime} /_{j=1}^{c_{3}} Y_{j}^{\prime} \nabla B(X, Y) /_{i=1}^{c_{2}} X_{i}^{\prime \prime} /_{j=1}^{c_{4}} Y_{j}^{\prime \prime}$.

In Figure 1, we give an illustrative example, where $|X|=3$ is a prime number and $|Y|=4=$ $2 \cdot 2$ is not. With respect to $B(X, Y) \cong B(X, Y) /_{i=1}^{c_{1}} X_{i}^{\prime} /_{j=1}^{c_{3}} Y_{j}^{\prime} \nabla B(X, Y) /_{i=1}^{c_{2}} X_{i}^{\prime \prime} /_{j=1}^{c_{4}} Y_{j}^{\prime \prime}$ in Figure 1, we have that $X_{1}^{\prime}=\left\{u_{1}\right\}, X_{2}^{\prime}=\left\{u_{2}\right\}, X_{3}^{\prime}=\left\{u_{3}\right\}, Y_{1}^{\prime}=\left\{v_{1}, v_{2}\right\}, Y_{2}^{\prime}=\left\{v_{3}, v_{4}\right\}, X_{1}^{\prime \prime}=\left\{u_{1}, u_{2}, u_{3}\right\}$, $Y_{1}^{\prime \prime}=\left\{v_{1}, v_{3}\right\}$ and $Y_{2}^{\prime \prime}=\left\{v_{2}, v_{4}\right\}$. In this example, we illustrate how we can decompose a complete bipartite graph $B(X, Y)$ where all the arcs have the same label. Because all arcs have the same label, the labels are omitted in all figures of this contribution.

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$B(X, Y) / X_{1}^{\prime} / X_{2}^{\prime} / X_{3}^{\prime} / Y_{1}^{\prime} / Y_{2}^{\prime}$ 区 $B(X, Y) / X_{1}^{\prime \prime} / Y_{1}^{\prime \prime} / Y_{2}^{\prime \prime}$
$B(X, Y) / X_{1}^{\prime} / X_{2}^{\prime} / X_{3}^{\prime} / Y_{1}^{\prime} / Y_{2}^{\prime}$


Figure 1. Decomposition of $B(X, Y) \cong B(X, Y) / X_{1}^{\prime} / X_{2}^{\prime} / X_{3}^{\prime} / Y_{1}^{\prime} / Y_{2}^{\prime} \boxtimes B(X, Y) / X_{1}^{\prime \prime} / Y_{1}^{\prime \prime} / Y_{2}^{\prime \prime}$. The set $\zeta$ from the proof of Lemma 3.1 and the graph isomorphic to $B(X, Y)$ induced by $\zeta$ in $B(X, Y) / X_{1}^{\prime} / X_{2}^{\prime} / X_{3}^{\prime} / Y_{1}^{\prime} / Y_{2}^{\prime} \boxtimes$ $B(X, Y) / X_{1}^{\prime \prime} / Y_{1}^{\prime \prime} / Y_{2}^{\prime \prime}$ is indicated within the dotted region (although not all arcs fit into this region).

Note that the proof of Lemma3.1 is modelled along the same lines as the proofs of the theorems presented in [4] and in [7].

Lemma 3.1. Let $B(X, Y)$ be a weakly connected complete bipartite graph where the labels of all arcs are the same. Let $[X, Y]$ contain only forward arcs, or let $[X, Y]$ contain only backward arcs. Let $|X|=c_{1} \cdot c_{2},|Y|=c_{3} \cdot c_{4}, c_{1}, \ldots, c_{4} \in \mathbb{N}$. Then there exist $X_{g}^{\prime}, X_{h}^{\prime \prime}, Y_{i}^{\prime}$ and $Y_{j}^{\prime \prime}$ such that $B(X, Y) \cong B(X, Y) /_{g=1}^{c_{1}} X_{g}^{\prime} /_{i=1}^{c_{3}} Y_{i}^{\prime} \boxtimes B(X, Y) \int_{h=1}^{c_{2}} X_{h}^{\prime \prime} /_{j=1}^{c_{4}} Y_{j}^{\prime \prime}$.

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Proof. Let $[X, Y]$ contain only forward arcs. It is sufficient to define a mapping $\phi: V(B(X, Y, Z))$ $\rightarrow V\left(B(X, Y) \int_{g=1}^{c_{1}} X_{g}^{\prime}{ }_{i=1}^{c_{3}} Y_{i}^{\prime} \boxtimes B(X, Y) \int_{h=1}^{c_{2}} X_{h}^{\prime \prime} \int_{j=1}^{c_{4}} Y_{j}^{\prime \prime}\right)$ and to prove that $\phi$ is an isomorphism from $B(X, Y)$ to $B(X, Y) \int_{g=1}^{c_{1}} X_{g}^{\prime} /_{i=1}^{c_{3}} Y_{i}^{\prime} \boxtimes B(X, Y, Z) \int_{h=1}^{c_{2}} X_{h}^{\prime \prime} \int_{j=1}^{c_{4}} Y_{j}^{\prime \prime}$.

Because $B(X, Y)$ is complete there are arcs from each vertex of $X$ to all vertices of $Y$. Then, without loss of generality, we define $X=\left\{u_{1,1}, \ldots, u_{1, c_{2}}, \ldots, u_{c_{1}, 1}, \ldots, u_{c_{1}, c_{2}}\right\}$. Now, we can contract $X$ using the sets $X_{1}^{\prime}, \ldots, X_{c_{1}}^{\prime}, X_{g}^{\prime}=\left\{u_{g, 1}, \ldots, u_{g, c_{2}}\right\},\left|X_{g}^{\prime}\right|=c_{2}, g=1, \ldots, c_{1}$. The vertices in the sets $X_{1}^{\prime}, \ldots, X_{c_{1}}^{\prime}$ are then replaced by the vertices $\tilde{x}_{1}^{\prime}, \ldots, \tilde{x}_{c_{1}}^{\prime}$, respectively, (note that there are no arcs that have both their ends in $X_{g}^{\prime}$ ), and we can contract $X$ using the sets $X_{1}^{\prime \prime}, \ldots, X_{c_{2}}^{\prime \prime}$, $X_{h}^{\prime \prime}=\left\{u_{1, h}, \ldots, u_{c_{1}, h}\right\},\left|X_{h}^{\prime \prime}\right|=c_{1}, h=1, \ldots, c_{2}$. The vertices in the sets $X_{1}^{\prime \prime}, \ldots, X_{c_{2}}^{\prime \prime}$ are then replaced by the vertices $\tilde{x}_{1}^{\prime \prime}, \ldots, \tilde{x}_{c_{2}}^{\prime \prime}$, respectively, (note that there are no arcs that have both their ends in $X_{h}^{\prime \prime}$ ). Likewise, for $|Y|=c_{3} \cdot c_{4}$, we define $Y=\left\{v_{1,1}, \ldots, v_{1, c_{4}}, \ldots, v_{c_{3}, 1}, \ldots, v_{c_{3}, c_{4}}\right\}$. Then, we can contract $Y$ using the sets $Y_{1}^{\prime}, \ldots, Y_{c_{3}}^{\prime}, Y_{i}^{\prime}=\left\{v_{i, 1}, \ldots, v_{i, c_{4}}\right\},\left|Y_{i}^{\prime}\right|=c_{4}, i=1, \ldots, c_{3}$. The vertices in the sets $Y_{1}^{\prime}, \ldots, Y_{c_{3}}^{\prime}$ are then replaced by the vertices $\tilde{y}_{1}^{\prime}, \ldots, \tilde{y}_{c_{3}}^{\prime}$, respectively, (note that there are no arcs that have both their ends in $Y_{k}^{\prime}$ ), and we can contract $Y$ using the sets $Y_{1}^{\prime \prime}, \ldots, Y_{c_{4}}^{\prime \prime}, Y_{j}^{\prime \prime}=\left\{v_{1, l}, \ldots, v_{c_{3}, j}\right\},\left|Y_{j}^{\prime}\right|=c_{3}, j=1, \ldots, c_{4}$. The vertices in the sets $Y_{1}^{\prime \prime}, \ldots, Y_{c_{4}}^{\prime \prime}$ are then replaced by the vertices $\tilde{y}_{1}^{\prime \prime}, \ldots, \tilde{y}_{c_{4}}^{\prime \prime}$, respectively, (note that there are no arcs that have both their ends in $Y_{j}^{\prime \prime}$ ).

Consider the mapping $\phi: V(B(X, Y)) \rightarrow V\left(B(X, Y){ }_{g=1}^{c_{1}} X_{g}^{\prime}{ }_{i=1}^{c_{3}} Y_{i}^{\prime} \boxtimes B(X, Y){ }_{h=1}^{c_{2}} X_{h}^{\prime \prime}{ }_{j=1}^{c_{4}} Y_{j}^{\prime \prime}\right)$ defined by $\phi\left(u_{g, h}\right)=\left(\tilde{x}_{g}^{\prime}, \tilde{x}_{h}^{\prime \prime}\right), \phi\left(v_{i, j}\right)=\left(\tilde{y}_{i}^{\prime}, \tilde{y}_{j}^{\prime \prime}\right)$. Then $\phi$ is obviously a bijective map if $V(B(X, Y)$ $\left.\int_{g=1}^{c_{1}} X_{g}^{\prime} \int_{i=1}^{c_{3}} Y_{i}^{\prime} \boxtimes B(X, Y) \int_{h=1}^{c_{2}} X_{h}^{\prime \prime} \int_{j=1}^{c_{4}} Y_{j}^{\prime \prime}\right)=\zeta$, where $\zeta$ is defined as $\zeta=\left\{\left(\tilde{x}_{g}^{\prime}, \tilde{x}_{h}^{\prime \prime}\right) \mid u_{g, h} \in X, \phi\left(u_{g, h}\right)=\right.$ $\left.\stackrel{g=1}{\left.g=\tilde{x}_{g}^{\prime}, \tilde{x}_{h}^{\prime \prime}\right)}\right\} \cup\left\{\left(\tilde{y}_{i}^{\prime}, \tilde{y}_{j}^{\prime \prime}\right) \mid v_{i, j} \in Y, \phi\left(v_{i, j}\right)=\left(\tilde{y}_{i}^{\prime}, \tilde{y}_{j}^{\prime \prime}\right)\right\}$. We are going to show this later by arguing all the other vertices (and their labelled arcs) $\left(\tilde{x}_{g}^{\prime}, \tilde{y}_{j}^{\prime \prime}\right),\left(\tilde{x}_{g}^{\prime}, \tilde{z}_{l}^{\prime \prime}\right),\left(\tilde{y}_{i}^{\prime}, \tilde{x}_{h}^{\prime \prime}\right)$ and $\left(\tilde{y}_{i}^{\prime}, \tilde{z}_{l}^{\prime \prime}\right)$ of $\left.B(X, Y)\right|_{g=1} ^{c_{1}} X_{g}^{\prime}{ }_{i=1}^{c_{3}} Y_{i}^{\prime}$ $\boxtimes B(X, Y) \int_{h=1}^{c_{2}} X_{h}^{\prime \prime} \int_{j=1}^{c_{4}} Y_{j}^{\prime \prime}$ will disappear from $B(X, Y) \int_{g=1}^{c_{1}} X_{g}^{\prime} \int_{i=1}^{c_{3}} Y_{i}^{\prime} \boxtimes B(X, Y) \int_{h=1}^{c_{2}} X_{h}^{\prime \prime} \int_{j=1}^{c_{4}} Y_{j}^{\prime \prime}$. But first we are going to prove the following claim.
Claim 3.1. The subgraph of $B(X, Y) \int_{g=1}^{c_{1}} X_{g}^{\prime} /_{i=1}^{c_{3}} Y_{i}^{\prime} \boxtimes B(X, Y, Z) \int_{h=1}^{c_{2}} X_{h}^{\prime \prime} \int_{j=1}^{c_{4}} Y_{j}^{\prime \prime}$ induced by $\zeta$ is isomorphic to $B(X, Y)$.

Proof. Firstly, $\phi$ is a bijection from $V(B(X, Y))$ to $\zeta$. Secondly, an arc $u_{i_{1}, i_{2}} v_{j_{1}, j_{2}}$ in $B(X, Y)$

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corresponds to the $\operatorname{arc} \tilde{x}_{i_{1}}^{\prime} \tilde{y}_{j_{1}}^{\prime}$ in $B(X, Y) \int_{i=1}^{c_{1}} X_{i}^{\prime} \int_{j=1}^{c_{3}} Y_{j}^{\prime}$ and to the $\operatorname{arc} \tilde{x}_{i_{2}}^{\prime \prime} \tilde{y}_{j_{2}}^{\prime \prime}$ in $B(X, Y) \int_{i=1}^{c_{2}} X_{i}^{\prime \prime} \int_{j=1}^{c_{4}} Y_{j}^{\prime \prime}$. Because all arcs of $\left.B(X, Y)\right|_{i=1} ^{c_{1}} X_{i}^{\prime} \int_{j=1}^{c_{3}} Y_{j}^{\prime}$ and $B(X, Y) \prod_{i=1}^{c_{2}} X_{i}^{\prime \prime} \int_{j=1}^{c_{4}} Y_{j}^{\prime \prime}$ are synchronising arcs, for each pair of arcs $\tilde{x}_{i_{1}}^{\prime} \tilde{y}_{j_{1}}^{\prime}, \tilde{x}_{i_{2}}^{\prime \prime} \tilde{y}_{j_{2}}^{\prime \prime}$ of $B(X, Y){ }_{i=1}^{c_{1}} X_{i}^{\prime} /_{j=1}^{c_{3}} Y_{j}^{\prime}$ and $B(X, Y){ }_{i=1}^{c_{2}} X_{i}^{\prime \prime}{ }_{j=1}^{c_{4}} Y_{j}^{\prime \prime}$, respectively, there is an $\operatorname{arc}\left(\tilde{x}_{i_{1}}^{\prime}, \tilde{x}_{i_{2}}^{\prime \prime}\right)\left(y_{j_{1}}^{\prime} \tilde{y}_{j_{2}}^{\prime \prime}\right)$ of $\left.B(X, Y)\right|_{i=1} ^{c_{1}} X_{i}^{\prime} \int_{j=1}^{c_{3}} Y_{j}^{\prime} \boxtimes B(X, Y){ }_{i=1}^{c_{2}} X_{i}^{\prime \prime} \int_{j=1}^{c_{4}} Y_{j}^{\prime \prime}$ and, therefore an $\operatorname{arc} u_{i_{1}, i_{2}} v_{j_{1}, j_{2}}$ in $B(X, Y)$ corresponds to an $\operatorname{arc}\left(\tilde{x}_{i_{1}}^{\prime}, \tilde{x}_{i_{2}}^{\prime \prime}\right)\left({\tilde{y^{\prime}}}_{i_{1}} \tilde{y}_{i_{2}}^{\prime \prime}\right)$ in $B(X, Y){ }_{i=1}^{c_{1}} X_{i}^{\prime}{ }_{j=1}^{c_{3}} Y_{j}^{\prime} \boxtimes$ $B(X, Y){ }_{i=1}^{c_{2}} X_{i}^{\prime \prime}{ }_{j=1}^{c_{4}} Y_{j}^{\prime \prime}$.

Hence, the map $\phi$ is a bijjection from $B(X, Y)$ to the subgraph of $B(X, Y){ }_{i=1}^{c_{1}} X_{i}^{\prime} /_{j=1}^{c_{3}} Y_{j}^{\prime} \boxtimes$ $B(X, Y) \int_{i=1}^{c_{2}} X_{i}^{\prime \prime} /_{j=1}^{c_{4}} Y_{j}^{\prime \prime}$ induced by $\zeta$ preserving the arcs and their labels and, therefore, $B(X, Y)$ is isomorphic to the subgraph of $B(X, Y) \int_{i=1}^{c_{1}} X_{i}^{\prime} /_{j=1}^{c_{3}} Y_{j}^{\prime} \boxtimes B(X, Y) \int_{i=1}^{c_{2}} X_{i}^{\prime \prime} \int_{j=1}^{c_{4}} Y_{j}^{\prime \prime}$ induced by $\zeta$. This completes the proof of Claim 3.1 .

We continue with the proof of Lemma3.1. It remains to show that all vertices of $V(B(X, Y)$ $\left.\int_{i=1}^{c_{1}} X_{i}^{\prime} \int_{j=1}^{c_{3}} Y_{j}^{\prime} \square B(X, Y) \int_{i=1}^{c_{2}} X_{i}^{\prime \prime} /_{j=1}^{c_{4}} Y_{j}^{\prime \prime}\right) \backslash \zeta$ (and the arcs of which these vertices are an end) disappear from $B(X, Y){ }_{i=1}^{c_{1}} X_{i}^{\prime} /_{j=1}^{c_{3}} Y_{j}^{\prime} \boxtimes B(X, Y){ }_{i=1}^{c_{2}} X_{i}^{\prime \prime} \int_{j=1}^{c_{4}} Y_{j}^{\prime \prime}$. This follows directly by the observation that only the vertices $\left(\tilde{x}_{i}^{\prime}, \tilde{x}_{j}^{\prime \prime}\right)$ have level 0 by definition of the Cartesian product. Then, all other vertices $\left(\tilde{x}_{i}^{\prime}, \tilde{y}_{j}^{\prime \prime}\right)$ and $\left(\tilde{y}_{i}^{\prime}, \tilde{x}_{j}^{\prime \prime}\right)$ have level $>0$ in $B(X, Y) \int_{i=1}^{c_{1}} X_{i}^{\prime} \int_{j=1}^{c_{2}} Y_{j}^{\prime} \square B(X, Y) \int_{i=1}^{c_{2}} X_{i}^{\prime \prime} \int_{j=1}^{c_{4}} Y_{j}^{\prime \prime}$. But these vertices $\left(\tilde{x}_{i}^{\prime}, \tilde{y}_{j}^{\prime \prime}\right)$ and $\left(\tilde{y}_{i}^{\prime}, \tilde{x}_{j}^{\prime \prime}\right)$ have level 0 in $B(X, Y) \prod_{i=1}^{c_{1}} X_{i}^{\prime} /\left._{j=1}^{c_{3}} Y_{j}^{\prime} \boxtimes B(X, Y)\right|_{i=1} ^{c_{2}} X_{i}^{\prime \prime} \int_{j=1}^{c_{4}} Y_{j}^{\prime \prime}$ and are, therefore, removed from $B(X, Y) \int_{i=1}^{c_{1}} X_{i}^{\prime} /_{j=1}^{c_{3}} Y_{j}^{\prime} \boxtimes B(X, Y){ }_{i=1}^{c_{2}} X_{i}^{\prime \prime} \int_{j=1}^{c_{4}} Y_{j}^{\prime \prime}$, together with the arcs of which these vertices are an end. This is because there are no $\operatorname{arcs} a$ with $\operatorname{head}(a)=\tilde{x}_{i}^{\prime}$ in $B(X, Y) \int_{i=1}^{c_{1}} X_{i}^{\prime} \int_{j=1}^{c_{3}} Y_{j}^{\prime}$, and, therefore, there are no arcs $b$ with head $(b)=\left(\tilde{x}_{i}^{\prime}, \tilde{y}_{j}^{\prime \prime}\right)$ in $B(X, Y)$ $\int_{i=1}^{c_{1}} X_{i}^{\prime} \int_{j=1}^{c_{3}} Y_{j}^{\prime} \boxtimes B(X, Y) \prod_{i=1}^{c_{2}} X_{i}^{\prime \prime} \int_{j=1}^{c_{4}} Y_{j}^{\prime \prime}$, and there are no $\operatorname{arcs} a$ with head $(a)=\tilde{x}_{i}^{\prime \prime}$ in $B(X, Y) \prod_{i=1}^{c_{2}} X_{i}^{\prime \prime}$ $\int_{j=1}^{c_{4}} Y_{j}^{\prime \prime}$ and, therefore, there are no arcs $b$ with $\operatorname{head}(b)=\left(\tilde{y}_{j}^{\prime}, \tilde{x}_{i}^{\prime \prime}\right)$ in $B(X, Y) \int_{i=1}^{c_{1}} X_{i}^{\prime} \int_{j=1}^{c_{3}} Y_{j}^{\prime}$ 区 $B(X, Y) \int_{i=1}^{c_{2}} X_{i}^{\prime \prime} \int_{j=1}^{c_{4}} Y_{j}^{\prime \prime}$. Hence, $\left(\tilde{x}_{i}^{\prime}, \tilde{y}_{j}^{\prime \prime}\right)$ and $\left(\tilde{y}_{j}^{\prime}, \tilde{x}_{i}^{\prime \prime}\right)$ must have level 0 in $B(X, Y) \int_{i=1}^{c_{1}} X_{i}^{\prime} \int_{j=1}^{c_{3}} Y_{i}^{\prime}$ 区

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$B(X, Y){ }_{i=1}^{c_{2}} X_{i}^{\prime \prime}{ }_{j=1}^{c_{4}} Y_{j}^{\prime \prime}$. Then $B(X, Y) \cong B(X, Y) \prod_{i=1}^{c_{1}} X_{i}^{\prime} \prod_{j=1}^{c_{3}} Y_{j}^{\prime} \boxtimes B(X, Y){ }_{i=1}^{c_{2}} X_{i}^{\prime \prime}{ }_{j=1}^{c_{4}} Y_{j}^{\prime \prime}$. The proof for $[X, Y]$ containing only backward arcs is similar. This completes the proof of Lemma3.1,

We continue with the decomposition of a 3 -partite graph $B(X, Y, Z)$. In Figure 2, we show an example of a 3-partite graph $B(X, Y, Z)$ that is decomposed in graphs $B(X, Y, Z) / X_{1}^{\prime} / Y_{1}^{\prime} / Y_{2}^{\prime} / Y_{3}^{\prime} /$ $Z_{1}^{\prime}$ and $B(X, Y, Z) / X_{1}^{\prime \prime} / Y_{1}^{\prime \prime} / Z_{1}^{\prime \prime} / Z_{2}^{\prime \prime}$. Note that the bipartite subgraph induced by the vertex set $Y \cup Z$ is not complete, but the bipartite subgraph arc-induced by the arcs of $[Y, Z]$ is complete.

$B(X, Y, Z) / X^{\prime} / Y_{1}^{\prime} / Y_{2}^{\prime} / Y_{3}^{\prime} / Z^{\prime}$

$B(X, Y, Z) / X^{\prime \prime} / Y^{\prime \prime} / Z_{1}^{\prime \prime} / Z_{2}^{\prime \prime}$

$B(X, Y, Z) / X^{\prime} / Y_{1}^{\prime} / Y_{2}^{\prime} / Y_{3}^{\prime} / Z^{\prime} \boxtimes B(X, Y, Z) / X^{\prime \prime} / Y^{\prime \prime} / Z_{1}^{\prime \prime} / Z_{2}^{\prime \prime}$


Figure 2. Decomposition of $B(X, Y, Z) \cong B(X, Y, Z) / X_{1}^{\prime} / Y_{1}^{\prime} / Y_{2}^{\prime} / Y_{3}^{\prime} / Z_{1}^{\prime} \boxtimes B(X, Y, Z) / X_{1}^{\prime \prime} / Y_{1}^{\prime \prime} / Z_{1}^{\prime \prime} / Z_{2}^{\prime \prime}, X_{1}^{\prime}=$ $\left\{u_{1}\right\}, Y_{1}^{\prime}=\left\{u_{2}\right\}, Y_{2}^{\prime}=\left\{u_{3}\right\}, Y_{3}^{\prime}=\left\{u_{4}\right\}, Z_{1}^{\prime}=\left\{u_{5}, u_{6}\right\}, X_{1}^{\prime \prime}=\left\{u_{1}\right\}, Y_{1}^{\prime \prime}=\left\{u_{2}, u_{3}, u_{4}\right\}, Z_{1}^{\prime \prime}=\left\{u_{5}\right\}, Z_{2}^{\prime \prime}=$ $\left\{u_{6}\right\}$. The set $\zeta$ and the graph isomorphic to $B(X, Y, Z)$ induced by $\zeta$ in $V\left(B(X, Y, Z) / X_{1}^{\prime} / Y_{1}^{\prime} / Y_{2}^{\prime} / Y_{3}^{\prime} / Z_{1}^{\prime}\right.$ 区 $\left.B(X, Y, Z) / X_{1}^{\prime \prime} / Y_{1}^{\prime \prime} / Z_{1}^{\prime \prime} / Z_{2}^{\prime \prime}\right)$ is indicated within the dotted region.

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Before we continue with Lemma 3.2 in which we decompose a 3-partite graph, we show in Figure 3 that if the set of vertices $W^{\prime \prime} \subset Y$ of the bipartite subgraph $B\left(W^{\prime \prime}, Z\right)$ of $B(X, Y, Z)$ arcinduced by the arcs of $[Y, Z]$ is not a subset of the set of vertices $W^{\prime} \subset Y$ of the bipartite subgraph $B\left(X, W^{\prime}\right)$ of $B(X, Y, Z)$ arc-induced by the arcs of $[X, Y]$, then this leads to the removal of the vertices $\left(\tilde{y}_{1}^{\prime}, \tilde{y}_{1}^{\prime \prime}\right)$ and $\left(\tilde{y}_{2}^{\prime}, \tilde{y}_{1}^{\prime \prime}\right)$ from $B(X, Y, Z) / X^{\prime} / Y_{1}^{\prime} / Y_{2}^{\prime} / Z^{\prime} \boxtimes B(X, Y, Z) / X^{\prime \prime} / Y_{1}^{\prime \prime} / Y_{2}^{\prime \prime} / Y_{3}^{\prime \prime} / Z^{\prime \prime}$ (at the lower right of Figure 3) giving $B(X, Y, Z) / X_{1}^{\prime} / Y_{1}^{\prime} / Y_{2}^{\prime} / Z_{1}^{\prime} \boxtimes B(X, Y, Z) / X_{1}^{\prime \prime} / Y_{1}^{\prime \prime} / Y_{2}^{\prime \prime} / Y_{3}^{\prime \prime} / Z_{1}^{\prime \prime}$ (at the lower left of Figure 3) and, therefore, $B(X, Y, Z) \not \equiv B(X, Y, Z) / X_{1}^{\prime} / Y_{1}^{\prime} / Y_{2}^{\prime} / Z_{1}^{\prime} \boxtimes B(X, Y, Z)$ $/ X_{1}^{\prime \prime} / Y_{1}^{\prime \prime} / Y_{2}^{\prime \prime} / Y_{3}^{\prime \prime} / Z_{1}^{\prime \prime}$. The vertices $\left(\tilde{y}_{1}, \tilde{y}_{1}^{\prime \prime}\right)$ and $\left(\tilde{y}_{2}, \tilde{y}_{1}^{\prime \prime}\right)$ are removed because they have level $>0$ in $B(X, Y, Z) / X_{1}^{\prime} / Y_{1}^{\prime} / Y_{2}^{\prime} / Z_{1}^{\prime} \square B(X, Y, Z) / X_{1}^{\prime \prime} / Y_{1}^{\prime \prime} / Y_{2}^{\prime \prime} / Y_{3}^{\prime \prime} / Z_{1}^{\prime \prime}$ and level 0 in $B(X, Y, Z) / X_{1}^{\prime} / Y_{1}^{\prime}$ $/ Y_{2}^{\prime} / Z_{1}^{\prime} \boxtimes B(X, Y, Z) / X_{1}^{\prime \prime} / Y_{1}^{\prime \prime} / Y_{2}^{\prime \prime} / Y_{3}^{\prime \prime} / Z_{1}^{\prime \prime}$.


Figure 3. The set of vertices $W^{\prime \prime}=\left\{u_{2}, u_{3}, u_{4}, u_{5}\right\}$ of the bipartite subgraph $B\left(W^{\prime \prime}, Z\right)$ of $B(X, Y, Z)$ arc-induced by the arcs of $[Y, Z]$ is not a subset of the set of vertices $W^{\prime}=\left\{u_{4}, u_{5}, u_{6}, u_{7}\right\}$ of the bipartite subgraph $B\left(X, W^{\prime}\right)$ of $B(X, Y, Z)$ arc-induced by the arcs of $[X, Y]$. Although, the graph induced by the set of vertices $\zeta$ is isomorphic to $B(X, Y, Z)$, we have that $B(X, Y, Z) \not \equiv B(X, Y, Z) / X_{1}^{\prime} / Y_{1}^{\prime} / Y_{2}^{\prime} / Z_{1}^{\prime} \triangle B(X, Y, Z) / X_{1}^{\prime \prime} / Y_{1}^{\prime \prime} / Y_{2}^{\prime \prime} / Y_{3}^{\prime \prime} / Z_{1}^{\prime \prime}$ (due to the removal of the vertices $\left(\tilde{y}_{1}^{\prime}, \tilde{y}_{1}^{\prime \prime}\right)$ and $\left.\left(\tilde{y}_{2}^{\prime}, \tilde{y}_{1}^{\prime \prime}\right)\right)$ by the VRSP.

Furthermore, in Figure 4, we show that when the requirement of Lemma 3.2 that $[X, Y]$ and

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$[Y, Z]$ contain the only arcs of $B(X, Y, Z)$ is violated, we have that the graph induced by the vertices of $\zeta$ is not isomorphic to $B(X, Y, Z)$. Hence, $B(X, Y, Z) \not \equiv B(X, Y, Z) / X_{1}^{\prime} / Y_{1}^{\prime} / Y_{2}^{\prime} / Z_{1}^{\prime} \boxtimes$ $B(X, Y, Z) / X_{1}^{\prime \prime} / Y_{1}^{\prime \prime} / Y_{2}^{\prime \prime} / Z_{1}^{\prime \prime}$. Note that $\left(\tilde{x}_{1}^{\prime}, \tilde{y}_{2}^{\prime \prime}\right),\left(\tilde{y}_{1}^{\prime}, \tilde{x}_{2}^{\prime \prime}\right),\left(\tilde{y}_{1}^{\prime}, \tilde{x}_{1}^{\prime \prime}\right),\left(\tilde{y}_{2}^{\prime}, \tilde{x}_{2}^{\prime \prime}\right)$ and $\left(\tilde{y}_{2}^{\prime}, \tilde{x}_{1}^{\prime \prime}\right)$, and the arcs of which $\left(\tilde{x}_{1}^{\prime}, \tilde{y}_{2}^{\prime \prime}\right),\left(\tilde{y}_{1}^{\prime}, \tilde{x}_{2}^{\prime \prime}\right),\left(\tilde{y}_{1}^{\prime}, \tilde{x}_{1}^{\prime \prime}\right),\left(\tilde{y}_{2}^{\prime}, \tilde{x}_{2}^{\prime \prime}\right)$ and $\left(\tilde{y}_{2}^{\prime}, \tilde{x}_{1}^{\prime \prime}\right)$ are an end are removed from $B(X, Y, Z)$ $/ X_{1}^{\prime} / Y_{1}^{\prime} / Y_{2}^{\prime} / Z_{1}^{\prime} \boxtimes B(X, Y, Z) / X_{1}^{\prime \prime} / Y_{1}^{\prime \prime} / Y_{2}^{\prime \prime} / Z_{1}^{\prime \prime}$ by the VRSP.


## $B(X, Y, Z) / X_{1}^{\prime \prime} / X_{2}^{\prime \prime} / Y_{1}^{\prime \prime} / Y_{2}^{\prime \prime} / Z_{1}^{\prime \prime}$


$B(X, Y, Z) / X_{1}^{\prime} / Y_{1}^{\prime} / Y_{2}^{\prime} / Z_{1}^{\prime} \boxtimes B(X, Y, Z) / X_{1}^{\prime \prime} / X_{2}^{\prime \prime} / Y_{1}^{\prime \prime} / Y_{2}^{\prime \prime} / Z_{1}^{\prime \prime}$


Figure 4. The requirement that $[X, Y]$ and $[Y, Z]$ contain the only arcs of $B(X, Y, Z)$ is violated, giving that the graph induced by the vertices of $\zeta$ is not isomorphic to $B(X, Y, Z)$. Hence, $B(X, Y, Z) \not \equiv$ $B(X, Y, Z) / X^{\prime} / Y_{1}^{\prime} / Y_{2}^{\prime} / Z^{\prime} \boxtimes B(X, Y, Z) / X^{\prime \prime} / Y_{1}^{\prime \prime} / Y_{2}^{\prime \prime} / Z^{\prime \prime}$. Furthermore, $\left(\tilde{x}_{1}^{\prime}, \tilde{y}_{2}^{\prime \prime}\right),\left(\tilde{y}_{1}^{\prime}, \tilde{x}_{2}^{\prime \prime}\right),\left(\tilde{y}_{1}^{\prime}, \tilde{x}_{1}^{\prime \prime}\right),\left(\tilde{y}_{2}^{\prime}, \tilde{x}_{2}^{\prime \prime}\right)$ and $\left(\tilde{y}_{2}^{\prime}, \tilde{x}_{1}^{\prime \prime}\right)$, and the $\operatorname{arcs}$ of which $\left(\tilde{x}_{1}^{\prime}, \tilde{y}_{2}^{\prime \prime}\right),\left(\tilde{y}_{1}^{\prime}, \tilde{x}_{2}^{\prime \prime}\right),\left(\tilde{y}_{1}^{\prime}, \tilde{x}_{1}^{\prime \prime}\right),\left(\tilde{y}_{2}^{\prime}, \tilde{x}_{2}^{\prime \prime}\right)$ and $\left(\tilde{y}_{2}^{\prime}, \tilde{x}_{1}^{\prime \prime}\right)$ are an end are removed from $B(X, Y, Z) / X^{\prime} / Y_{1}^{\prime} / Y_{2}^{\prime} / Z^{\prime} \boxtimes B(X, Y, Z) / X^{\prime \prime} / Y_{1}^{\prime \prime} / Y_{2}^{\prime \prime} / Z^{\prime \prime}$ by the VRSP.

The requirement that $[X, Y]$ and $[Y, Z]$ contain only forward arcs, or $[X, Y]$ and $[Y, Z]$ contain only backward arcs must not be violated. Otherwise, for example, if $[X, Y]$ contains only

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forward arcs and $[Y, Z]$ contains only backward arcs, we have a bipartite graph $B(X \cup Z, Y)$ where $[X \cup Z, Y]$ contains only forward arcs. Because such a graph is not complete it is also not decomposable by Lemma 6.1 in [3] as is shown by the example given in Figure 4 in [3]. Now, for the decomposition of a 3-partite graph $B(X, Y, Z)$, we have the following lemma.

Lemma 3.2. Let $B(X, Y, Z)$ be a weakly connected 3-partite graph where the labels of all arcs are the same. Let $[X, Y]$ and $[Y, Z]$ contain the only arcs of $B(X, Y, Z)$. Let $[X, Y]$ and $[Y, Z]$ contain only forward arcs, or let $[X, Y]$ and $[Y, Z]$ contain only backward arcs. Let $B(X, Y)$ be a complete bipartite subgraph of $B(X, Y, Z)$ induced by $X \cup Y$. Let $B(W, Z), W \subseteq Y$, be a complete bipartite subgraph of $B(X, Y, Z)$ arc-induced by all arcs of $[Y, Z]$. Let $|X|=c_{1} \cdot c_{2}, G C D(|Y|,|W|)=$ $c_{3},|Y|=c_{3} \cdot c_{4},|W|=c_{3} \cdot c_{7},|Z|=c_{5} \cdot c_{6}, c_{1}, \ldots, c_{7} \in \mathbb{N}$. Then there exist $X_{g}^{\prime}, X_{h}^{\prime \prime}, Y_{i}^{\prime}, Y_{j}^{\prime \prime}, Z_{k}^{\prime}, Z_{l}^{\prime \prime}$, such that $B(X, Y, Z) \cong B(X, Y, Z) /_{g=1}^{c_{1}} X_{g}^{\prime} \int_{i=1}^{c_{3}} Y_{i}^{\prime} \int_{k=1}^{c_{5}} Z_{k}^{\prime} \boxtimes B(X, Y, Z) /_{h=1}^{c_{2}} X_{h}^{\prime \prime} \int_{j=1}^{c_{4}} Y_{j}^{\prime \prime} /_{l=1}^{c_{6}} Z_{l}^{\prime \prime}$.

Proof. Let $[X, Y]$ and $[Y, Z]$ contain only forward arcs. It is sufficient to define a mapping $\phi$ : $V(B(X, Y, Z)) \rightarrow V\left(B(X, Y, Z) \int_{g=1}^{c_{1}} X_{g}^{\prime} /_{i=1}^{c_{3}} Y_{i}^{\prime} \int_{k=1}^{c_{5}} Z_{k}^{\prime} \boxtimes B(X, Y, Z) \prod_{h=1}^{c_{2}} X_{h}^{\prime \prime} \int_{j=1}^{c_{4}} Y_{j}^{\prime \prime} \int_{l=1}^{c_{6}} Z_{l}^{\prime \prime}\right)$ and to prove that $\phi$ is an isomorphism from $B(X, Y, Z)$ to $B(X, Y, Z) \int_{g=1}^{c_{1}} X_{g}^{\prime} /_{i=1}^{c_{3}} Y_{i}^{\prime} \int_{k=1}^{c_{5}} Z_{k}^{\prime} \boxtimes B(X, Y, Z){ }_{h=1}^{c_{2}} X_{h}^{\prime \prime}$ ${ }_{j=1}^{c_{4}} Y_{j}^{\prime \prime}{ }_{l=1}^{c_{6}} Z_{l}^{\prime \prime}$.

Because, $|X|=c_{1} \cdot c_{2}$, we define $X=\left\{u_{1,1}, \ldots, u_{1, c_{2}}, \ldots, u_{c_{1}, 1}, \ldots, u_{c_{1}, c_{2}}\right\}$. Then, we can contract $X$ using the sets $X_{1}^{\prime}, \ldots, X_{c_{1}}^{\prime}, X_{g}^{\prime}=\left\{u_{g, 1}, \ldots, u_{g, c_{2}}\right\},\left|X_{g}^{\prime}\right|=c_{2}, g=1, \ldots, c_{1}$. The vertices in the sets $X_{1}^{\prime}, \ldots, X_{c_{1}}^{\prime}$ are then replaced by the vertices $\tilde{x}_{1}^{\prime}, \ldots, \tilde{x}_{c_{1}}^{\prime}$, respectively, (note that there are no arcs that have both their ends in $X_{g}^{\prime}$ ), and we can contract $X$ using the sets $X_{1}^{\prime \prime}, \ldots, X_{c_{2}}^{\prime \prime}$, $X_{h}^{\prime \prime}=\left\{u_{1, h}, \ldots, u_{c_{1}, h}\right\},\left|X_{h}^{\prime \prime}\right|=c_{1}, h=1, \ldots, c_{2}$. The vertices in the sets $X_{1}^{\prime \prime}, \ldots, X_{c_{2}}^{\prime \prime}$ are then replaced by the vertices $\tilde{x}_{1}^{\prime \prime}, \ldots, \tilde{x}_{c_{2}}^{\prime \prime}$, respectively, (note that there are no arcs that have both their ends in $X_{h}^{\prime \prime}$ ).

Because, $|Y|=c_{3} \cdot c_{4},|W|=c_{3} \cdot c_{7}$ and $W \subseteq Y$ we have that $c_{7} \leqslant c_{4}$. Therefore, we define $Y=\left\{v_{1,1}, \ldots, v_{1, c_{4}}, \ldots, v_{c_{3}, 1}, \ldots, v_{c_{3}, c_{4}}\right\}$ and $W=\left\{v_{1,1}, \ldots, v_{1, c_{7}}, \ldots, v_{c_{3}, 1}, \ldots, v_{c_{3}, c_{7}}\right\}$, satisfying $W \subseteq Y$. Then, we can contract $Y$ using the sets $Y_{1}^{\prime}, \ldots, Y_{c_{3}}^{\prime}, Y_{i}^{\prime}=\left\{v_{i, 1}, \ldots, v_{i, c_{4}}\right\},\left|Y_{i}^{\prime}\right|=$ $c_{4}, i=1, \ldots, c_{3}$. The vertices in the sets $Y_{1}^{\prime}, \ldots, Y_{c_{3}}^{\prime}$ are then replaced by the vertices $\tilde{y}_{1}^{\prime}, \ldots, \tilde{y}_{c_{3}}^{\prime}$, respectively, (note that there are no arcs that have both their ends in $Y_{i}^{\prime}$ ), and we can contract $Y$

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using the sets $Y_{1}^{\prime \prime}, \ldots, Y_{c_{4}}^{\prime \prime}, Y_{j}^{\prime \prime}=\left\{v_{1, j}, \ldots, v_{c_{3}, j}\right\},\left|Y_{j}^{\prime \prime}\right|=c_{3}, j=1, \ldots, c_{4}$. The vertices in the sets $Y_{1}^{\prime \prime}, \ldots, Y_{c_{4}}^{\prime \prime}$ are then replaced by the vertices $\tilde{y}_{1}^{\prime \prime}, \ldots, \tilde{y}_{c_{4}}^{\prime \prime}$, respectively, (note that there are no arcs that have both their ends in $Y_{h}^{\prime}$ ). Using analogue definitions for $W_{i}^{\prime}$ and $W_{j}^{\prime \prime}$ with respect to the definitions of $Y_{i}^{\prime}$ and $Y_{j}^{\prime \prime}$, we have that the vertices in the sets $W_{1}^{\prime}, \ldots, W_{c_{3}}^{\prime}$ are replaced by the vertices $\tilde{y}_{1}^{\prime}, \ldots, \tilde{y}_{c_{3}}^{\prime}$ and the vertices in the sets $W_{1}^{\prime \prime}, \ldots, W_{c_{7}}^{\prime \prime}$ are replaced by the vertices $\tilde{y}_{1}^{\prime \prime}, \ldots, \tilde{y}_{c_{7}}^{\prime \prime}$.

Because, $|Z|=c_{5} \cdot c_{6}$, we define $Z=\left\{w_{1,1}, \ldots, w_{1, c_{6}}, \ldots, w_{c_{5}, 1}, \ldots, w_{c_{5}, c_{6}}\right\}$. Then, we can contract $Z$ using the sets $Z_{1}^{\prime}, \ldots, Z_{c_{5}}^{\prime}, Z_{k}^{\prime}=\left\{w_{k, 1}, \ldots, w_{k, c_{6}}\right\},\left|Z_{k}^{\prime}\right|=c_{6}, k=1, \ldots, c_{5}$. The vertices in the sets $Z_{1}^{\prime}, \ldots, Z_{c_{5}}^{\prime}$ are then replaced by the vertices $\tilde{z}_{1}^{\prime}, \ldots, \tilde{z}_{c_{5}}^{\prime}$, respectively, (note that there are no arcs that have both their ends in $Z_{k}^{\prime}$ ), and we can contract $Z$ using the sets $Z_{1}^{\prime \prime}, \ldots, Z_{c_{6}}^{\prime \prime}$, $Z_{l}^{\prime \prime}=\left\{w_{1, l}, \ldots, w_{c_{5}, l}\right\},\left|Z_{l}^{\prime}\right|=c_{5}, l=1, \ldots, c_{6}$. The vertices in the sets $Z_{1}^{\prime \prime}, \ldots, Z_{c_{6}}^{\prime \prime}$ are then replaced by the vertices $\tilde{z}_{1}^{\prime \prime}, \ldots, \tilde{z}_{c_{6}}^{\prime \prime}$, respectively, (note that there are no arcs that have both their ends in $Z_{l}^{\prime \prime}$ ).

Consider the mapping $\phi: V(B(X, Y, Z)) \rightarrow V\left(B(X, Y, Z) /_{g=1}^{c_{1}} X_{g}^{\prime}{ }_{i=1}^{c_{3}} Y_{i}^{\prime}{ }_{k=1}^{c_{5}} Z_{k}^{\prime} \boxtimes B(X, Y, Z)\right.$ $\left.\int_{h=1}^{c_{2}} X_{h}^{\prime \prime} \int_{j=1}^{c_{4}} Y_{j}^{\prime \prime} \int_{l=1}^{c_{6}} Z_{l}^{\prime \prime}\right)$ defined by $\phi\left(u_{g, h}\right)=\left(\tilde{x}_{g}^{\prime}, \tilde{x}_{h}^{\prime \prime}\right), \phi\left(v_{i, j}\right)=\left(\tilde{y}_{i}^{\prime}, \tilde{y}_{j}^{\prime \prime}\right), \phi\left(w_{k, l}\right)=\left(\tilde{z}_{k}^{\prime}, \tilde{z}_{l}^{\prime \prime}\right)$. Then $\phi$ is obviously a bijective map if $V\left(B(X, Y, Z) \int_{g=1}^{c_{1}} X_{g}^{\prime} \int_{i=1}^{c_{3}} Y_{i}^{\prime} \int_{k=1}^{c_{5}} Z_{k}^{\prime} \nabla B(X, Y, Z){ }_{h=1}^{c_{2}} X_{h}^{\prime \prime} \int_{j=1}^{c_{4}} Y_{j}^{\prime \prime} \int_{l=1}^{c_{6}} Z_{l}^{\prime \prime}\right)=\zeta$, where $\zeta$ is defined as $\zeta=\left\{\left(\tilde{x}_{g}^{\prime}, \tilde{x}_{h}^{\prime \prime}\right) \mid u_{g, h} \in X, \phi\left(u_{g, h}\right)=\left(\tilde{x}_{g}^{\prime}, \tilde{x}_{h}^{\prime \prime}\right)\right\} \cup\left\{\left(\tilde{y}_{i}^{\prime}, \tilde{y}_{j}^{\prime \prime}\right) \mid v_{i, j} \in Y, \phi\left(v_{i, j}\right)=\right.$ $\left.\left(\tilde{y}_{i}^{\prime}, \tilde{y}_{j}^{\prime \prime}\right)\right\} \cup\left\{\left(\tilde{z}_{k}^{\prime}, \tilde{z}_{l}^{\prime \prime}\right) \mid w_{k, l} \in Z, \phi\left(w_{k, l}\right)=\left(\tilde{z}_{k}^{\prime}, \tilde{z}_{l}^{\prime \prime}\right)\right\}$.

We are going to show this later by arguing all the other vertices (and their labelled arcs) $\left(\tilde{x}_{g}^{\prime}, \tilde{y}_{j}^{\prime \prime}\right),\left(\tilde{x}_{g}^{\prime}, \tilde{z}_{l}^{\prime \prime}\right),\left(\tilde{y}_{i}^{\prime}, \tilde{x}_{h}^{\prime \prime}\right),\left(\tilde{y}_{i}^{\prime}, \tilde{z}_{l}^{\prime \prime}\right),\left(\tilde{z}_{k}^{\prime}, \tilde{x}_{h}^{\prime \prime}\right)$ and $\left(\tilde{z}_{k}^{\prime}, \tilde{y}_{j}^{\prime \prime}\right)$ of $B(X, Y, Z) /_{g=1}^{c_{1}} X_{g}^{\prime} /_{i=1}^{c_{3}} Y_{i}^{\prime} /_{k=1}^{c_{5}} Z_{k}^{\prime} \boxtimes B(X, Y$, $Z) \int_{h=1}^{c_{2}} X_{h}^{\prime \prime} \int_{j=1}^{c_{4}} Y_{j}^{\prime \prime} \int_{l=1}^{c_{6}} Z_{l}^{\prime \prime}$ will disappear from $B(X, Y, Z){ }_{g=1}^{c_{1}} X_{g}^{\prime} /\left._{i=1}^{c_{3}} Y_{i}^{\prime} \int_{k=1}^{c_{5}} Z_{k}^{\prime} \boxtimes B(X, Y, Z)\right|_{h=1} ^{c_{2}} X_{h}^{\prime \prime} \int_{j=1}^{c_{4}} Y_{j}^{\prime \prime}$ ${ }_{l=1}^{c_{6}} Z_{l}^{\prime \prime}$. But first we are going to prove the following claim.
Claim 3.2. The subgraph of $B(X, Y, Z) \int_{g=1}^{c_{1}} X_{g}^{\prime} \int_{i=1}^{c_{3}} Y_{i}^{\prime} \int_{k=1}^{c_{5}} Z_{k}^{\prime} \boxtimes B(X, Y, Z) \int_{h=1}^{c_{2}} X_{h}^{\prime \prime} \int_{j=1}^{c_{4}} Y_{j}^{\prime \prime} \int_{l=1}^{c_{6}} Z_{l}^{\prime \prime}$ induced by $\zeta$ is isomorphic to $B(X, Y, Z)$.

Proof. $\phi$ is a bijection from $V(B(X, Y, Z))$ to $\zeta$. It remains to show that this bijection preserves the arcs and their label pairs. Due to Lemma 3.1 and because the subgraph $B(X, Y)$ is a complete bipartite subgraph, we have that an arc $u_{g, h} v_{i, j}$ in $[X, Y]$ is represented by the arc

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$\left(\tilde{x}_{g}^{\prime}, \tilde{x}_{h}^{\prime \prime}\right)\left(\tilde{y}_{i}^{\prime}, \tilde{y}_{j}^{\prime \prime}\right)$ in $A\left(B(X, Y, Z) \prod_{g=1}^{c_{1}} X_{g}^{\prime} \int_{i=1}^{c_{3}} Y_{i}^{\prime} \sum_{k=1}^{c_{5}} Z_{k}^{\prime} \boxtimes B(X, Y, Z){ }_{h=1}^{c_{2}} X_{h}^{\prime \prime} \int_{j=1}^{c_{4}} Y_{j}^{\prime \prime}{ }_{l=1}^{c_{6}} Z_{l}^{\prime \prime}\right)$ and, due to Lemma 3.1 and because the subgraph $B(W, Z)$ is a complete bipartite subgraph, an arc $v_{i, j} w_{k, l}$ in $[W, Z]$ is represented by the $\operatorname{arc}\left(\tilde{y}_{i}^{\prime}, \tilde{y}_{j}^{\prime \prime}\right)\left(\tilde{z}_{k}^{\prime}, \tilde{z}_{l}^{\prime \prime}\right)$ in $A\left(B(X, Y, Z){ }_{g=1}^{c_{1}} X_{g}^{\prime} /{ }_{i=1}^{c_{3}} Y_{i}^{\prime} /_{k=1}^{c_{5}} Z_{k}^{\prime} \boxtimes B(X, Y, Z)\right.$ $\left.\int_{h=1}^{c_{2}} X_{h}^{\prime \prime} \int_{j=1}^{c_{4}} Y_{j}^{\prime \prime} /_{l=1}^{c_{6}} Z_{l}^{\prime \prime}\right)$. Together with $[Y \backslash W, Z]=\varnothing$, we have that the subgraph of $B(X, Y, Z){ }_{g=1}^{c_{1}} X_{g}^{\prime}$ $\int_{i=1}^{c_{3}} Y_{i}^{\prime} \int_{k=1}^{c_{5}} Z_{k}^{\prime} \boxtimes B(X, Y, Z) \int_{h=1}^{c_{2}} X_{h}^{\prime \prime} \int_{j=1}^{c_{4}} Y_{j}^{\prime \prime}{ }_{l=1}^{c_{6}} Z_{l}^{\prime \prime}$ induced by $\zeta$ is isomorphic to $B(X, Y, Z)$.

We continue with the proof of Lemma 3.2. It remains to show that all other vertices of $B(X, Y, Z){ }_{g=1}^{c_{1}} X_{g}^{\prime} \int_{i=1}^{c_{3}} Y_{i}^{\prime} \int_{k=1}^{c_{5}} Z_{k}^{\prime} \boxtimes B(X, Y, Z){ }_{h=1}^{c_{2}} X_{h}^{\prime \prime}{ }_{j=1}^{c_{4}} Y_{j}^{\prime \prime}{ }_{l=1}^{c_{6}} Z_{l}^{\prime \prime}$, except for the vertices of $\zeta$, disappear from $\left.B(X, Y, Z)\right|_{g=1} ^{c_{1}} X_{g}^{\prime} \int_{i=1}^{c_{3}} Y_{i}^{\prime} \int_{k=1}^{c_{5}} Z_{k}^{\prime} \boxtimes B(X, Y, Z){ }_{h=1}^{c_{2}} X_{h}^{\prime \prime}{ }_{j=1}^{c_{4}} Y_{j}^{\prime \prime}{ }_{l=1}^{c_{6}} Z_{l}^{\prime \prime}$. Due to Lemma 3.1, this is clear for the vertices $\left(\tilde{x}_{g}^{\prime}, \tilde{y}_{j}^{\prime \prime}\right)$ and $\left(\tilde{y}_{i}^{\prime}, \tilde{x}_{h}^{\prime \prime}\right)$. Likewise, due to Lemma 3.1 and the removal of all $\left(\tilde{x}_{g}^{\prime}, \tilde{y}_{j}^{\prime \prime}\right)$ and $\left(\tilde{y}_{i}^{\prime}, \tilde{x}_{h}^{\prime \prime}\right)$ ), this is also clear for the vertices $\left(\tilde{y}_{i}^{\prime}, \tilde{z}_{l}^{\prime \prime}\right)$ and $\left(\tilde{z}_{k}^{\prime}, \tilde{y}_{j}^{\prime \prime}\right)$. Remains to show that the vertices $\left(\tilde{x}_{g}^{\prime}, \tilde{z}_{l}^{\prime \prime}\right)$ and $\left(\tilde{z}_{k}^{\prime}, \tilde{x}_{h}^{\prime \prime}\right)$ are also removed from $B(X, Y, Z) \int_{g=1}^{c_{1}} X_{g}^{\prime} \int_{i=1}^{c_{3}} Y_{i}^{\prime} \int_{k=1}^{c_{5}} Z_{k}^{\prime}$ 区 $B(X, Y, Z) \int_{h=1}^{c_{2}} X_{h}^{\prime \prime} /_{j=1}^{c_{4}} Y_{j}^{\prime \prime}{ }_{l=1}^{c_{6}} Z_{l}^{\prime \prime}$. Because there are no $\operatorname{arcs} a$ with head $(a)=\tilde{x}_{g}^{\prime}$ in $B(X, Y, Z){ }_{g=1}^{c_{1}} X_{g}^{\prime}$ $\int_{i=1}^{c_{3}} Y_{i}^{\prime}{ }_{k=1}^{c_{5}} Z_{k}^{\prime}$ there are no arcs $b$ with head $(b)=\left(\tilde{x}_{g}^{\prime}, \tilde{z}_{l}^{\prime \prime}\right)$ in $B(X, Y, Z){ }_{g=1}^{c_{1}} X_{g}^{\prime} \int_{i=1}^{c_{3}} Y_{i}^{\prime} \prod_{k=1}^{c_{5}} Z_{k}^{\prime} \boxtimes B(X, Y$, Z) $\int_{h=1}^{c_{2}} X_{h}^{\prime \prime} \int_{j=1}^{c_{4}} Y_{j}^{\prime \prime}{ }_{l=1}^{c_{6}} Z_{l}^{\prime \prime}$, and, because there are no arcs $a$ with $h e a d(a)=\tilde{x}_{h}^{\prime \prime}$ in $B(X, Y, Z) \int_{h=1}^{c_{2}} X_{h}^{\prime \prime} \int_{j=1}^{c_{4}}$ $Y_{j}^{\prime \prime} /_{l=1}^{c_{6}} Z_{l}^{\prime \prime}$ there are no $\operatorname{arcs} b$ with $h e a d(b)=\left(\tilde{z}_{k}^{\prime}, \tilde{x}_{h}^{\prime \prime}\right)$ in $\left.\left.B(X, Y, Z)\right|_{g=1} ^{c_{1}} X_{g}^{\prime}{ }_{i=1}^{c_{3}} Y_{i}^{\prime}\right|_{k=1} ^{c_{5}} Z_{k}^{\prime} \boxtimes B(X, Y, Z)$ $\int_{h=1}^{c_{2}} X_{h}^{\prime \prime} \int_{j=1}^{c_{4}} Y_{j}^{\prime \prime}{ }_{l=1}^{c_{6}} Z_{l}^{\prime \prime}$. Hence, the $\left(x_{g}^{\prime}, z_{l}^{\prime \prime}\right)$ and $\left(\tilde{z}_{k}^{\prime}, \tilde{x}_{h}^{\prime \prime}\right)$ must have level 0 in $B(X, Y, Z) \int_{g=1}^{c_{1}} X_{g}^{\prime}{ }_{i=1}^{c_{3}} Y_{i}^{\prime}$ $\int_{k=1}^{c_{5}} Z_{k}^{\prime} \boxtimes B(X, Y, Z) \int_{h=1}^{c_{2}} X_{h}^{\prime \prime} \int_{j=1}^{c_{4}} Y_{j}^{\prime \prime} \int_{l=1}^{c_{6}} Z_{l}^{\prime \prime}$. But, the level of $\left(\tilde{x}_{g}^{\prime}, \tilde{z}_{l}^{\prime \prime}\right)$ and $\left(\tilde{z}_{k}^{\prime}, \tilde{x}_{h}^{\prime \prime}\right)$ is greater than zero in $B(X, Y, Z) \prod_{g=1}^{c_{1}} X_{g}^{\prime} \prod_{i=1}^{c_{3}} Y_{i}^{\prime} \prod_{k=1}^{c_{5}} Z_{k}^{\prime} \square B(X, Y, Z){ }_{h=1}^{c_{2}} X_{h}^{\prime \prime} \prod_{j=1}^{c_{4}} Y_{j}^{\prime \prime} \int_{l=1}^{c_{6}} Z_{l}^{\prime \prime}$ and the level of $\left(\tilde{x}_{g}^{\prime}, \tilde{z}_{l}^{\prime \prime}\right)$ and $\left(\tilde{z}_{k}^{\prime}, \tilde{x}_{h}^{\prime \prime}\right)$ is zero in $\left.\left.B(X, Y, Z)\right|_{g=1} ^{c_{1}} X_{g}^{\prime} \int_{i=1}^{c_{3}} Y_{i}^{\prime} \int_{k=1}^{c_{5}} Z_{k}^{\prime} \boxtimes B(X, Y, Z)\right|_{h=1} ^{c_{2}} X_{h}^{\prime \prime} \int_{j=1}^{c_{4}} Y_{j}^{\prime \prime}{ }_{l=1}^{c_{6}} Z_{l}^{\prime \prime}$. Therefore, $\left(\tilde{x}_{g}^{\prime}, \tilde{z}_{l}^{\prime \prime}\right)$ and $\left(\tilde{z}_{k}^{\prime}, \tilde{x}_{h}^{\prime \prime}\right)$ are removed from $\left.B(X, Y, Z)\right|_{g=1} ^{c_{1}} X_{g}^{\prime} \int_{i=1}^{c_{3}} Y_{i}^{\prime} /_{k=1}^{c_{5}} Z_{k}^{\prime} \boxtimes B(X, Y, Z){ }_{h=1}^{c_{2}} X_{h}^{\prime \prime} \int_{j=1}^{c_{4}} Y_{j}^{\prime \prime}{ }_{l=1}^{c_{6}} Z_{l}^{\prime \prime}$. Because there are no other vertices in $B(X, Y, Z) \int_{g=1}^{c_{1}} X_{g}^{\prime} \int_{i=1}^{c_{3}} Y_{i}^{\prime} \int_{k=1}^{c_{5}} Z_{k}^{\prime} \boxtimes B(X, Y, Z) \int_{h=1}^{c_{2}} X_{h}^{\prime \prime} \int_{j=1}^{c_{4}} Y_{j}^{\prime \prime} \int_{l=1}^{c_{6}} Z_{l}^{\prime \prime}$, we have that $B(X, Y, Z) \cong B(X, Y, Z) \int_{g=1}^{c_{1}} X_{g}^{\prime} \int_{i=1}^{c_{3}} Y_{i}^{\prime} \int_{k=1}^{c_{5}} Z_{k}^{\prime} \boxtimes B(X, Y, Z) \prod_{h=1}^{c_{2}} X_{h}^{\prime \prime} \int_{j=1}^{c_{4}} Y_{j}^{\prime \prime} \int_{l=1}^{c_{6}} Z_{l}^{\prime \prime}$. The

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proof for $[X, Y]$ and $[Y, Z]$ containing only backward arcs is similar. This completes the proof of Lemma 3.2,

Note, if $c_{1}=c_{3}=c_{5}=1$ then we have for $B(X, Y, Z)$ that each vertex $u_{g, h}$ of $X$ corresponds to a vertex $\tilde{x}_{g, h}^{\prime \prime}$ of $B(X, Y, Z) \prod_{h=1}^{c_{2}} X_{h}^{\prime \prime} \int_{j=1}^{c_{4}} Y_{j}^{\prime \prime} \int_{l=1}^{c_{6}} Z_{l}^{\prime \prime}$, the vertex $v_{i, j}$ of $Y$ corresponds to the vertex $\tilde{y}_{i, j}^{\prime \prime}$ of $\left.B(X, Y, Z)\right|_{h=1} ^{c_{2}} X_{h}^{\prime \prime} \int_{j=1}^{c_{4}} Y_{j}^{\prime \prime} \int_{l=1}^{c_{6}} Z_{l}^{\prime \prime}$, the vertex $w_{k, l}$ of $Z$ corresponds to the vertex $\tilde{z}_{k, l}^{\prime \prime}$ of $B(X, Y, Z){ }_{h=1}^{c_{2}} X_{h}^{\prime \prime} \int_{j=1}^{c_{4}} Y_{j}^{\prime \prime} \int_{l=1}^{c_{6}} Z_{l}^{\prime \prime}$, each arc $u_{g, h} v_{i, j}$ corresponds to an $\operatorname{arc} \tilde{x}_{g, h}^{\prime \prime} \tilde{y}_{i, j}^{\prime \prime}$ of $B(X, Y, Z){ }_{h=1}^{c_{2}} X_{h}^{\prime \prime}$ $\int_{j=1}^{c_{4}} Y_{j}^{\prime \prime} \int_{l=1}^{c_{6}} Z_{l}^{\prime \prime}$ and each arc $v_{i, j} w_{k, l}$ corresponds to an $\operatorname{arc} \tilde{y}_{i, j}^{\prime \prime} \tilde{z}_{k, l}^{\prime \prime}$ of $B(X, Y, Z) \int_{h=1}^{c_{2}} X_{h}^{\prime \prime} \int_{j=1}^{c_{4}} Y_{j}^{\prime \prime} \int_{l=1}^{c_{6}} Z_{l}^{\prime \prime}$. This gives $B(X, Y, Z) \cong B(X, Y, Z) \int_{h=1}^{c_{2}} X_{h}^{\prime \prime} \int_{j=1}^{c_{4}} Y_{j}^{\prime \prime} \int_{l=1}^{c_{6}} Z_{l}^{\prime \prime}$. Furthermore, we have for $B(X, Y, Z)$ that the vertices of $X$ correspond to the vertex $\tilde{x}^{\prime}$ of $\left.B(X, Y, Z)\right|_{i=1} ^{c_{1}} X_{i}^{\prime} \int_{i=1}^{c_{3}} Y_{i}^{\prime}{ }_{i=1}^{c_{5}} Z_{i}^{\prime}$, the vertices of $Y$ of $B(X, Y, Z) /_{i=1}^{c_{1}} X_{i}^{\prime} /_{i=1}^{c_{3}} Y_{i}^{\prime} /_{i=1}^{c_{5}} Z_{i}^{\prime}$ correspond to the vertex $\tilde{y}^{\prime}$, the vertices of $Z$ correspond to the vertex $\tilde{z}^{\prime}$ of $\left.B(X, Y, Z)\right|_{i=1} ^{c_{1}} X_{i}^{\prime} \prod_{i=1}^{c_{3}} Y_{i}^{\prime} \prod_{i=1}^{c_{5}} Z_{i}^{\prime}$, each arc $u_{g, h} v_{i, j}$ corresponds to the arc $\tilde{x}^{\prime} \tilde{y}^{\prime}$ of $B(X, Y, Z){ }_{h=1}^{c_{1}} X_{h}^{\prime} \int_{j=1}^{c_{3}} Y_{j}^{\prime} \int_{l=1}^{c_{5}} Z_{l}^{\prime}$ and each arc $v_{i, j} w_{k, l}$ corresponds to the $\operatorname{arc} \tilde{y}^{\prime} \tilde{z}^{\prime}$ of $B(X, Y, Z)$ $\int_{h=1}^{c_{1}} X_{h}^{\prime} \int_{j=1}^{c_{3}} Y_{j}^{\prime} \int_{l=1}^{c_{5}} Z_{l}^{\prime}$. Then, $B(X, Y, Z) \int_{i=1}^{c_{1}} X_{i}^{\prime} \int_{i=1}^{c_{3}} Y_{i}^{\prime} \int_{i=1}^{c_{5}} Z_{i}^{\prime}$ is a path from $\tilde{x}^{\prime}$ to $\tilde{z}^{\prime}$ and we have that $\left.\left.B(X, Y, Z) \cong B(X, Y, Z)\right|_{i=1} ^{c_{1}} X_{i}^{\prime} \int_{i=1}^{c_{3}} Y_{i}^{\prime} \int_{i=1}^{c_{5}} Z_{i}^{\prime} \boxtimes B(X, Y, Z){ }_{i=1}^{c_{2}} X_{i}^{\prime \prime}\right|_{i=1} ^{c_{4}} Y_{i}^{\prime \prime}{ }_{i=1}^{c_{6}} Z_{i}^{\prime \prime}$. Because there is no reduction of the number of vertices (and arcs) in $B(X, Y, Z){ }_{h=1}^{c_{2}} X_{h}^{\prime \prime} \int_{j=1}^{c_{4}} Y_{j}^{\prime \prime}{ }_{l=1}^{c_{6}} Z_{l}^{\prime \prime}$ with respect to $B(X, Y, Z)$, this is a useless decomposition. Likewise, for $c_{2}=c_{4}=c_{6}=1$, we have as well such a useless decomposition. Therefore, at least one of the values of $G C D\left(c_{1}, c_{2}\right), G C D\left(c_{3}, c_{4}\right)$ or $G C D\left(c_{5}, c_{6}\right)$ has to be greater than one, or, in case $|X|,|Y|$ and $|Z|$ are prime numbers, at least one but not all of the $c_{1}, c_{3}, c_{5}$ have to be greater than one (and, therefore, at least one but not all of the $c_{2}, c_{4}, c_{6}$ is greater than one).

We continue with the decomposition of an $n$-partite graph $B\left(X_{1}, \ldots, X_{n}\right)$ where all arcs have the same label, the arcs in $\left[X_{1}, X_{2}\right], \ldots,\left[X_{n-1}, X_{n}\right]$ are the only arcs of $B\left(X_{1}, \ldots, X_{n}\right)$, the subgraph $B\left(X_{1}, X_{2}\right)$ of $B\left(X_{1}, \ldots, X_{n}\right)$ induced by $X_{1} \cup X_{2}$ is a complete bipartite graph, each subgraph $B\left(X_{i}, \chi_{i+1}\right), i=2, \ldots, n-1$, of $B\left(X_{1}, \ldots, X_{n}\right)$ arc induced by the arcs of $\left[X_{i}, X_{i+1}\right]$ is

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a complete bipartite graph with $\chi_{i+1} \subseteq X_{i+1}$ (note that $\chi_{n}=X_{n}$ ). The partition of $X_{1}$ in subsets of $X_{1}$ is similar to the partition of $X$ in subsets of $X$ in Lemma3.2, the partition of each of the $X_{2}, \ldots, X_{n-1}$ in subsets of $X_{2}, \ldots, X_{n-1}$ is similar to the partition of $Y$ in subsets of $Y$ in Lemma 3.2, the partition of each of the $\chi_{2}, \ldots, \chi_{n-1}$ in subsets of $\chi_{2}, \ldots, \chi_{n-1}$ is similar to the partition of $W$ in subsets of $W$ in Lemma 3.2 and the partition of $X_{n}$ in subsets of $X_{n}$ is similar to the partition of $Z$ in subsets of $Z$ in Lemma3.2. Following these requirements, we state and prove the following decomposition theorem.

Theorem 3.1. Let $B\left(X_{1}, \ldots, X_{n}\right)$ be a weakly connected $n$-partite graph where the labels of all arcs are the same. Let $\left[X_{1}, X_{2}\right], \ldots,\left[X_{n-1}, X_{n}\right]$ contain the only arcs of $B\left(X_{1}, \ldots, X_{n}\right)$. Let $\left[X_{i}, X_{i+1}\right]$ contain only forward arcs for all $i \in\{1, \ldots, n-1\}$ or let $\left[X_{i}, X_{i+1}\right]$ contain only backward arcs for all $i \in\{1, \ldots, n-1\}$. Let $c_{1,1}, c_{1,2}, c_{2,3}, \ldots, c_{n-1,3}, c_{2,4}, \ldots, c_{n-1,4}, c_{n, 5}, c_{n, 6}, c_{2,7}, \ldots$, $c_{n-1,7} \in \mathbb{N}$. Let $\left|X_{1}\right|=c_{1,1} \cdot c_{1,2}$. Let $B\left(X_{1}, X_{2}\right)$ be the complete bipartite subgraph of $B\left(X_{1}, \ldots\right.$, $\left.X_{n}\right)$ induced by $X_{1} \cup X_{2}$. Let $B\left(\chi_{i}, X_{i+1}\right), \chi_{i} \subseteq X_{i}, i \in\{2, \ldots, n-1\}$ be the complete bipartite subgraph of $B\left(X_{1}, \ldots, X_{n}\right)$ arc-induced by the $\operatorname{arcs}$ in $\left[X_{i}, X_{i+1}\right]$. Let $G C D\left(\left|X_{i}\right|,\left|\chi_{i}\right|\right)=c_{i, 3}$, $\left|X_{i}\right|=c_{i, 3} \cdot c_{i, 4},\left|\chi_{i}\right|=c_{i, 3} \cdot c_{i, 7}, i \in\{2, \ldots, n-1\}$. Let $\left|X_{n}\right|=c_{n, 5} \cdot c_{n, 6}$. Then there exist $X_{1, g}^{\prime}, X_{m, i}^{\prime}, X_{n, k}^{\prime}, X_{1, h}^{\prime \prime}, X_{m, j}^{\prime \prime}, X_{n, l}^{\prime \prime}$ such that $B\left(X_{1}, \ldots, X_{n}\right) \cong B\left(X_{1}, \ldots, X_{n}\right) /_{g=1}^{c_{1,1}} X_{1, g}^{\prime} /_{m=2}^{n-1} /_{i=1}^{c_{i, 3}} X_{m, i}^{\prime}$ $/_{k=1}^{c_{n, 5}} X_{n, k}^{\prime} \nabla B\left(X_{1}, \ldots, X_{n}\right) /_{h=1}^{c_{1,2}} X_{1, h}^{\prime \prime} \int_{m=2}^{n-1} /_{j=1}^{c_{j, 4}} X_{m, j}^{\prime \prime} /_{l=1}^{c_{n, 6}} X_{n, l}^{\prime \prime}$.

Proof. Proof by induction. For $n=2$, we apply Lemma 3.1, For $n=3$, we apply Lemma 3.2, Let $B\left(X_{1}, \ldots, X_{n-1}\right)$ be decomposed into two $(n-1)$-partite graphs $B\left(X_{1}, \ldots, X_{n-1}\right) /_{g=1}^{c_{1,1}} X_{1, g}^{\prime} /_{m=2}^{n-2} l_{i=1}^{c_{i, 3}}$ $X_{m, i}^{\prime} \int_{k=1}^{c_{n-1,5}} X_{n-1, k}^{\prime}$ and $B\left(X_{1}, \ldots, X_{n-1}\right) /_{h=1}^{c_{1,2}} X_{1, h}^{\prime \prime} \int_{m=2}^{n-2} \int_{j=1}^{c_{j, 4}} X_{m, j}^{\prime \prime} /_{l=1}^{c_{6}} X_{n-1, l}^{\prime \prime}$, such that $B\left(X_{1}, \ldots, X_{n-1}\right)$ $\cong B\left(X_{1}, X_{2}, \ldots, X_{n-1}\right) /_{g=1}^{c_{1,1}} X_{1, g}^{\prime} \int_{m=2}^{n-2} /_{i=1}^{c_{i, 3}} X_{m, i}^{\prime} \int_{k=1}^{c_{n-1,5}} X_{n-1, k}^{\prime} \nabla B\left(X_{1}, \ldots, X_{n-1}\right) /_{h=1}^{c_{1,2}} X_{1, h}^{\prime \prime} /_{m=2}^{n-2} /_{j=1}^{c_{i, 4}} X_{m, j}^{\prime \prime}$ $\int_{l=1}^{c_{n-1,6}} X_{n-1, l}^{\prime \prime}$. Then, for $B\left(X_{1}, \ldots, X_{n}\right)$, we have the partition of $X_{n-1}$ in the sets $X_{n-1, i}^{\prime}, i=$ $1, \ldots c_{n-1,3}$ and the partition of $X_{n-1}$ in the sets $X_{n-1, j}^{\prime \prime}, j=1, \ldots c_{n-1,4}$, the partition of $\chi_{n-1} \subseteq$ $X_{n-1}$ in the sets $\chi_{n-1, i}^{\prime}, i=1, \ldots c_{n-1,3}$ and the partition of $\chi_{n-1}$ in the sets $\chi_{n-1, j}^{\prime \prime}, j=1, \ldots c_{n-1,7}$, and the partition of $X_{n}$ in the sets $X_{n, k}^{\prime}, k=1, \ldots c_{n, 5}$ and the partition of $X_{n}$ in the sets $X_{n, l}^{\prime \prime}, l=$ $1, \ldots c_{n, 6}$. Now, with similar arguments as for $Y$ and $Z$ of Lemma 3.2, we have that $B\left(X_{1}, \ldots, X_{n}\right)$

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$\cong B\left(X_{1}, \ldots, X_{n}\right) /_{g=1}^{c_{1,1}} X_{1, g}^{\prime} \int_{m=2}^{n-1} \int_{i=1}^{c_{i, 3}} X_{m, i}^{\prime} \int_{k=1}^{c_{n, 6}} X_{n, k}^{\prime} \nabla B\left(X, \ldots, X_{n}\right)_{h=1}^{c_{1,2}} X_{1, h}^{\prime \prime}{ }_{m=2}^{n-1} /_{j=1}^{c_{j, 4}} X_{m, j}^{\prime \prime} /_{l=1}^{c_{n}, 7} X_{n, l}^{\prime \prime}$. This completes the proof of Theorem 3.1.

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