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Living at a Singularity:

Field and String Theory on Orbifolds

vorgelegt von

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Abstract

In this review we describe the recent progress in understanding of field theory on singular orbifolds and compare some of these results with corresponding string computations. Using orbifold trace formulae expressions for anomalies and tadpoles are obtained. For more complicated computations like the renormalization of gauge multiplets orbifold compatible field theory is developed. The gap between orbifolds and smooth compactifications is bridged by constructing explicit orbifold blowups and toric orbifold resolutions.

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Chapter 1

General Introduction

Over the last thirty year a massive theoretical effort has been made to understand nature at its most fundamental level. The Standard Model of particle physics was formulated at the beginning of this era. This theory describing three generations of quarks and leptons, that interact via electroweak and color gauge interactions described by the groups SU(2), U(1) and SU(3), respectively, and has subsequently been tested in great depth. To date the Standard Model has been confirmed by all experiments to a very high level of precision, only the Higgs sector of the theory has not been directly observed yet, but no convincing hints for physics beyond this framework have been found.

Not withstanding these successes, the Standard Model is not consider to be the final –fundamental– theory of nature. Unless the Higgs mass is finely tuned its validity breaks down long before we reach the Planck scale at which quantum gravity effects become important. The real problem here is that this tuning has to be adjusted at each order in perturbation theory by an infinite amount. This in itself is not theoretically impossible, but it leaves a quadratic sensitivity to any high scale masses, which makes it difficult to understand why the electroweak breaking scale remains so low. One of the most prominent candidates to resolve this hierarchy problem is supersymmetry. This is a symmetry of a different class then ordinary gauge symmetries because it mixes bosons and fermions, i.e. particles with different spin statistics, with each other. The hierarchy problem is resolved because the contributions of the bosonic and fermionic states cancel enough to weaken the fine tuning sufficiently. This is sometimes referred to as the technical solution to the hierarchy problem, because it explain why two such different scales can exist side by side within one theory, but it does not explain why they are so different. More surprisingly if one follows the evolution of the gauge couplings with the energy scale, one finds that at a scale not to far off from the Planck scale the couplings meet. This is a strong hint for unification of the gauge interactions of the Standard Model, which means that at high energies the structure of the theory becomes easier; describe by one instead of three gauge couplings.

Supersymmetry has more remarkable consequences. Because spin of particles are properties of the Lorentz group, supersymmetry is a spacetime symmetry, rather than an internal symmetry. This becomes particularly apparent when one assumes that supersymmetry is a gauge symmetry itself: The resulting theory necessarily contains gravity, and therefore goes under the name supergravity. Normal gravity is a very difficult theory to quantize, because it is highly non–renormalizable, i.e. at each order in perturbation theory many new counter terms are need to make sense of it. Much like supersymmetry introduces a partial cancellation that stabilized the gauge hierarchy, supergravity leads to many restrictions greatly reducing the number of counter terms. There is a maximum on the number of supersymmetries a supergravity theory can have in order that the graviton remains a massless spin–

two particle that can be consistently coupled to other fields. Such maximal supergravities have the property that their renormalization is much more under control: It has been proven to be three loop finite [1–4]. Maximal supergravities can exist in at most 11 spacetime dimensions [5]. This is quite remarkable because ordinary field theories can in principle be formulated in any number of dimensions. Another striking feature in this story is that the scale of unification and the scale of (super)gravity, are quite close together. This might hint at an even bigger possible unification, in which both gauge interactions and gravity are combined into one single theory.

A theory, which does just that, has been invent: String theory. However, it requires the departure of the concept of point particles. Before it is was generally assumed that the basic building blocks of nature themselves do not have any internal structure. In this new theory one dimensional extended objects, strings, are the fundamental dynamical degrees of freedom. Their quantization requires studying conformal field theories that live on the world sheet, that the string sweeps out when it moves in time through space. To obtain a well–defined quantization one runs into various consistency requirements: The world sheet theories have to be (partly) supersymmetric, and the number of spacetime dimensions to be ten. This is close the bound on the number of dimensions found for supergravity, which was 11. Moreover the limit of small string tension, string theory can be described as a ten dimensional supergravity theory coupled to a super Yang–Mills gauge theory. The appearance of this gauge theory makes it possible that string theory can both unify gravity and the gauge interactions of the Standard Model.

When we say that all forces including gravity unify we ignore one major issue: In string theory this unification happens in ten dimensions, while the Standard Model, describing our world, only lives in four dimensions. A couple of ways have been invented to hide these six of the dimensions from our sight. The first idea is to make these extra dimensions so small, that they are invisible. Their presence could only be inferred by observing very massive particles maybe far beyond experimental reach. But this simple version of compactification is not good enough: Even if we cannot see these additional dimensions directly, some of their properties still remain. In particular, the four dimensional theory would be $N = 4$ supersymmetric (having four supercharges in four dimension) so that a chiral spectrum of quarks and leptons cannot arise. Such chiral spectra can only arise in theories with $N = 1$ supersymmetry or less. To still solve the hierarchy problem, one needs to require precisely $N = 1$ supersymmetry. A certain amount of supersymmetry can be broken by the compactification manifold we use in the internal dimensions. To achieve the breaking from $N = 4$ to $N = 1$ supersymmetry in four dimensions the compactification space has to be a six dimensional Calabi–Yau manifold.

Such Calabi–Yau manifolds are unfortunately rather complicated spaces. Their structure is so difficult that it is unknown how to construct explicit metrics on them. To nevertheless study resulting four dimensional theory that arise from compactifications on Calabi–Yau manifolds, a few approaches have been developed. The first one relies on the observation that many properties of the four dimensional theory do not depend very sensitive on the precise form of the metric, but are determined by the topological data of the Calabi–Yau only. This idea has been quite successful leading to models quite close to the supersymmetric extension of the Standard Model and is studied until this date. But since it requires a lot of abstract mathematics, it is not very clear what is exactly physically going on. A second approach to study Calabi–Yau manifolds is to construct spaces which are on the one hand almost like flat space, while on the other hand still have the crucial property that they break 1/4 of the original ten dimensional supersymmetry. These spaces, called orbifolds, are obtained by taking a flat space, say a torus, and divide out some finite order discrete symmetry. This leads to orbifold projections which throw out all states of the ten dimensional theory, that are not invariant under the discrete symmetry. The main advantage of orbifolds is that, besides their construction is rather easy, strings can be quantize exactly on them. This means that one can compute the full (not only massless) spectrum of the string. For field theories on orbifolds this is very different: Discrete symmetries have some fixed points. At these singular points things like the curvature of the space diverges, which in principle invalidates a field theoretical description. One reflection of this is that in field theory one cannot tell which additional states live at the orbifold fixed points. Here string theory does much better: An additional consistency requirement of strings at the one loop level, called modular invariance, implies that this low energy description is complete. As apposed to field theory models one cannot add by hand new fields to the low energy description of string theory, they are all predicted. As a consequence string theory predicts precisely which twisted string states live at the orbifold fixed points. All this shows that one can learn a lot about string theory compactifications using orbifolds.

Another way to hide some of the dimensions of string theory is to assume that string theory, in addition to strings, contains other dynamical objects of arbitrary dimensions. Such branes are the end points of open strings. On such branes gauge theories arise. Moreover, strings that stretch between different sets of branes that intersect can give rise to chiral matter. The simplest description of such branes is also singular because they are taken to be infinitely thin. Hence from a field theoretical point of view they suffer from similar problems as orbifold singularities. In fact many properties and branes and orbifold singularities are related via string dualities, maps between different regimes of a string theory or even different string theories. The full implications of the web of dualities is still being uncovered.

1.1 Overview of Topics

After this general overview of physics between the Standard Model and string theory, we explain in more depth what we review in this work. As discussed in the general introduction orbifolds are very useful objects to get insight into the properties of complicated Calabi–Yau manifolds. Besides considering orbifolds as simple toy settings for string theory, they can be studied in field theory. But as was mentioned above one has to be rather careful in doing so, because orbifolds have singularities that might take one beyond the validity of field theory. It is of course very interesting to study precise whether, when, and how the field theory description breaks down. In the cases where field theory does run into problems, it is worthwhile to see how string theory is able to cure these difficulties.

There are many issues that one can study to get insight into the regime which might just be beyond field theory reach and where string theory needs to kick in. In this review I only cover a few selected topics on which I have performed detailed analysis in previous and on going research projects. These studies can be divided into basically four different categories defined by the following keywords: consistency, stability, perturbativity, and deformations. To each of these topics a section is devoted in this work. Before going into details, let us introduce each of them:

The first question, addressed in section 3, is whether field theories on orbifold can make sense even in principle, i.e. whether they give rise to consistent field theories. The basic obstruction to consistency is the presence of anomalies. An anomaly arises when a symmetry, that the classical theory possesses, is lost in the process of quantization. For global symmetries this is not a severe problem, it simply means that the quantum theory is more complicated than its classical approximation. For gauge theories this is deadly because the numbers of degrees of theory that describes at classical and quantum levels are different. The global constraints of anomaly cancellation, i.e. that anomalies cancel at the effective zero mode level, have been often studied in the past, but the question whether anomalies really need to cancel locally (at the fixed points) is quite recent. In particular, one may wonder whether a model can have local anomalies as long as the zero mode theory is anomaly free. In addition, one may speculate that once one has local anomaly cancellation, the bulk anomalies are irrelevant. To investigate such questions we first describe how we can compute the local structure of anomalies on orbifolds. These computations show that bulk and local fixed points anomalies arise simultaneously. To argue that all bulk and fixed point localized anomalies need to cancel separately, we demonstrate that this always happens in modular invariant heterotic string models on orbifolds.

After having established the anomaly consistency requirements that theories on orbifolds need to fulfill, we ask in section 4 whether such configurations are stable. One direction one take this question is: What the consequences are of the curvature singularities at the fixed points? To shed some light on this issue in a simpler setting, we instead investigate what kind of gauge field tadpoles can arise at orbifold fixed points. We explain that tadpoles for the internal parts of gauge fields in higher dimensional theories are closely related to Fayet–Ilopolous D–term tadpoles in four dimensional supersymmetric field theories. We describe how these tadpoles can be calculated directly, and show that they arise generically both in simple field theory models with a few extra dimensions as well as in full low energy string effective field theory. We discuss that these divergent gauge field tadpoles could lead to strong localization of bulk states. We explain that this singular behavior might be an artifact of the field theory regularization. To this end we recompute the local gauge field tadpoles directly within the heterotic string context, showing that the singular behavior is smoothed out in a Gaussian fashion around the singularity.

The next question, whether we can do perturbative calculations on orbifolds, is address in section 5. As a concrete test case we investigate the behavior of gauge couplings on orbifolds. Even a simple Abelian gauge theory on a five dimensional orbifold has various gauge couplings, namely bulk and local fixed point gauge couplings. All of them can be renormalized by quantum corrections. In additions in the bulk extra higher derivative operators can be generated when the number of extra dimensions is larger than two. We consider these effects in detail and describe how they can be computed using a technique we call orbifold compatible field theory. We explain how these results can be extend to application to heterotic orbifold models and how they are related to previously considered string threshold corrections to gauge couplings.

The final issue we discuss in section 6 is how to resolve or blow up of orbifold singularities, i.e. deform them to become non–singular spaces. Orbifolds are special points in the full moduli space of the heterotic string on Calabi–Yau manifolds. In order to have control on the theory away from these special points, it is crucial to have a better understanding of model building on the corresponding smooth compactification spaces. A concrete way to probe the moduli space surrounding orbifold points is to consider blowups of orbifold singularities. The construction of explicit blowups is unfortunately not easy. The best known example is the Eguchi–Hanson resolution of the $\mathbb{C}^2/\mathbb{Z}_2$ orbifold singularity. We construct explicit blowups of $\mathbb{C}^n/\mathbb{Z}_n$ orbifolds with U(1) gauge bundles. The singularities of more complicated orbifolds might not allow for simple explicit blowup constructions. On the other hand, the topological properties of such resolutions can be conveniently described by toric geometry. We show that with these techniques we are able to reconstruct large classes of orbifold models in blowup.

Before we are able to discuss all these topics in detail, we first introduce the necessary concepts to describe to various theories in section 2. It gives on overview of preliminary material needed in this review. Since all the discussions will take place in the context of supersymmetric field theories, we first describe them in various dimensions first. Moreover, as we will be constantly comparing field theory results with those obtained using string theory, we need to collect their basic properties that relevant for the subsequent discussions. This means in particular that we will focus on the heterotic string which is mainly used in this work. Finally, we need a detailed overview orbifolds, both from a purely geometrical point of view as well as using field and string theory descriptions of them. This section and all others are concluded by a short summary of the main findings.

1.2 Used material

As this work is part of my cumulative habilitation, I list the various publications on which the sections of this work have been based:

Section 2 Preliminary Material

S. Groot Nibbelink and M. Hillenbach "Renormalization of supersymmetric gauge theories on orbifolds: Brane gauge couplings and higher derivative operators" *Phys. Lett.* **B616** (2005) 125–134 [`hep-th/0503153`].

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S. Groot Nibbelink, H. P. Nilles, M. Olechowski and M. G. A. Walter, "Localized tadpoles of anomalous heterotic U(1)'s," Nucl. Phys. B **665** (2003) 236 [arXiv:hep-th/0303101].

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Section 4 Stability: Tadpoles

D. M. Ghilencea, S. Groot Nibbelink, and H. P. Nilles "Gauge corrections and FI-term in 5D KK theories" *Nucl. Phys.* **B619** (2001) 385–395 [hep-th/0108184].

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Chapter 2

Preliminary Material

This section gives an introduction to the necessary material to understand the subsequent discussions in this work. We begin by introducing supersymmetric field theories in four and more dimensions. Next we briefly describe the heterotic string, and conclude this technical introduction with a review of orbifolds.

2.1 Supersymmetric Field Theories

Supersymmetry is a fast research field, so that a complete review would require a lengthy set of lectures. As there are many detailed textbooks and lecture notes on supersymmetry available (see e.g. [6–10]), we do not wish to give a concise overview here. Instead the aim of this subsection is merely to remind the reader of those aspects of supersymmetry that will be relevant for a basic understanding of the rest of this review. We have divided the presentation of the material into three parts. First we describe four dimensional supersymmetry using the familiar superfields. Next we consider both five and six dimensional supersymmetric theory together, by grouping the components in four dimensional superfields. Finally we consider the ten dimensional super Yang–Mills theory.

Supersymmetry in Four Dimensions

Of all supersymmetric theories in various dimensions the four dimensional case is best understood and developed in most detail. To describe most supersymmetric theories in four dimensions only three different multiplets are required: the chiral, vector, and supergravity multiplets. In this work we mainly work with the first two, hence we recall them in some detail.

The chiral multiplet contains a complex scalar z and a chiral fermion ψ_α on–shell. To make use of the convenient superfield notation also an auxiliary complex field F has to be introduced. The chiral superfield is defined by the superspace constraint $\overline{D}_{\dot\alpha}\Phi = 0$, and its components are obtained by the following restrictions

$$\Phi| \;=\; z \;, \qquad \frac{1}{\sqrt{2}}\, D_\alpha \Phi| \;=\; \psi_\alpha \;, \qquad -\frac{1}{4}\, D^2 \Phi^2| \;=\; F \;, \tag{2.1}$$

where $|$ denotes that all θ and $\bar\theta$ are put to zero after all supercovariant differentiations are performed. In a similar fashion the vector multiplet can be introduced: On–shell it describes a massless gauge field A_m and a Majorana fermion λ_α. These components are contained in the real superfield $V^\dagger = V$ as

$$\frac{1}{2}[D_\alpha, \overline{D}_{\dot\alpha}]V| = \sigma^m_{\alpha\dot\alpha} A_m , \qquad -\frac{1}{4}\overline{D}^2 D_\alpha V| = i\lambda_\alpha , \qquad \frac{1}{8}D^\alpha \overline{D}^2 D_\alpha V| = D , \tag{2.2}$$

where the auxiliary real field D is required by off–shell supersymmetry. The components listed here are the only ones present in Wess–Zumino gauge. In other gauges, like the supersymmetric Feynman gauge employed below, additional components may appear. The non–Abelian vector superfields are algebra valued functions, i.e. they can be decomposed as $V = V^i T_i$, where T_i are the Hermitian generators of some Lie group G. The corresponding Lie algebra $[T_i, T_j] = f_{ij}{}^k T_k$ is defined in terms of the purely imaginary structure coefficients $f_{ij}{}^k$. The coupling of a chiral multiplet to a vector multiplet is determined by the action

$$S_{C+V} = \int \mathrm{d}^4x \mathrm{d}^4\theta\, \overline{\Phi} e^{2V}\Phi + \int \mathrm{d}^4x \mathrm{d}^2\theta\, \frac{1}{4g^2}\mathrm{tr}\, W^\alpha W_\alpha + \int \mathrm{d}^4x \mathrm{d}^2\bar\theta\, \frac{1}{4g^2}\mathrm{tr}\, \overline{W}_{\dot\alpha}\overline{W}^{\dot\alpha} , \tag{2.3}$$

where we have introduced the superfield strength of the vector superfield

$$W_\alpha = -\frac{1}{8}\overline{D}^2\Big(e^{-2V}D_\alpha e^{2V}\Big) , \qquad W_\alpha| = i\lambda_\alpha , \qquad D_\beta W_\alpha| = -i(\sigma^{mn}\epsilon)_{\beta\alpha}F_{mn} - \epsilon_{\beta\alpha}D , \tag{2.4}$$

with F_{mn} is the (non–)Abelian field strength. This action is invariant under the super gauge transformations

$$\Phi \to e^{2\Lambda}\Phi , \qquad e^{2V} \to e^{2\overline{\Lambda}}e^{2V}e^{2\Lambda} , \qquad W_\alpha \to e^{-2\Lambda}W_\alpha e^{2\Lambda} , \tag{2.5}$$

where Λ denotes an arbitrary chiral superfield.

Supersymmetry in Five and Six Dimensions

The structure of five and six dimensional supersymmetric theories are very similar, because these theories form representations of essentially the same supersymmetry algebra. This means that we can discuss hyper and vector multiplets in five and six dimensions in parallel. Here we only focus on five or six dimensional $N = 1$ supersymmetry, which are $N = 2$ theories from the four dimensional point of view. To make maximally use of the convenient superspace language, we describe both multiplets in terms of four dimensional chiral and vector superfields introduced above. This method has been suggested by [11–13].

The hyper multiplet $\mathbb{H}$ describes four real scalars called hyperons, and a Dirac fermion in five dimensions. This hyperino turns into a chiral fermion in six. This field content shows that it can be described by two chiral superfields Φ_+ and Φ_-. In terms of the superfield language the action for this multiplet reads in six dimensions

$$S_{\mathbb{H}} = \int \mathrm{d}^6x \Big\{ \int \mathrm{d}^4\theta \left(\overline{\Phi}_+\Phi_+ + \Phi_-\overline{\Phi}_-\right) + \int \mathrm{d}^2\theta\, \Phi_-\,\partial\,\Phi_+ + \int \mathrm{d}^2\bar\theta\, \overline{\Phi}_+\,\bar\partial\,\overline{\Phi}_- \Big\} , \tag{2.6}$$

where $\partial = \partial_5 + i\,\partial_6$. In five dimensions only ∂_5 should be kept in the derivative ∂ and $\bar\partial$.

The vector multiplet $\mathbb{V}$ contains a six dimensional gauge field A_M, and a gaugino which is has a chirality opposite to that of the hyperino, because the six dimensional supersymmetry generator is

Figure 2.1: The propagators for a hyper multiplet $\mathbb{H} = (\Phi_+, \Phi_-)$ and a vector multiplet $\mathbb{V} = (V, S, C, C')$ (including ghosts) are displayed.

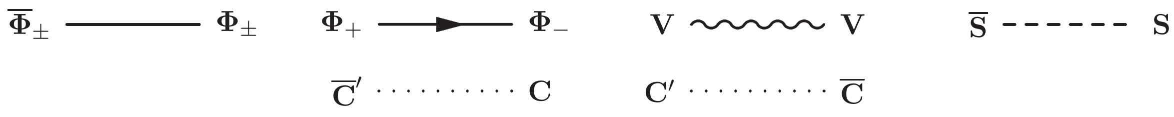
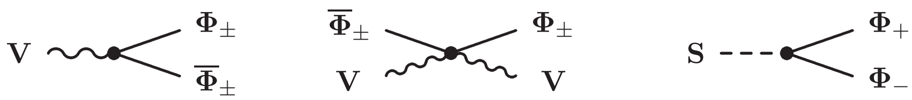


Figure 2.2: These interactions encode the coupling of the gauge multiplet to the hyper multiplet.



chiral itself. The field content in five dimensions can be understood via a straightforward Kaluza–Klein reduction and consists of a five dimensional gauge field, a real scalar and a Dirac fermion. The action in the Abelian case is given by

$$S_{\mathbb{V}} \;=\; \frac{1}{g^2}\int \mathrm{d}^6 x \Big\{ \int \mathrm{d}^2\theta \, \frac{1}{4} W^\alpha W_\alpha \;+\; \int \mathrm{d}^2\bar{\theta}\, \frac{1}{4} \overline{W}_{\dot{\alpha}} \overline{W}^{\dot{\alpha}} \;+\; \int \mathrm{d}^4\theta \left(\overline{S} S \;+\; \bar{\partial} V \partial V \right) \Big\} \,. \tag{2.7}$$

Supersymmetry in Ten Dimensions

Ten dimensional supersymmetry is very restrictive, hence only very few multiplets do exist: the vector and supergravity multiplets. As we will not be interested in supergravity effects, we only focus on the vector multiplet. It consists of a ten dimensional vector field A_M and a Majorana–Weyl gaugino. In terms of four dimensional superfields this multiplet can be decomposed into a vector superfield V and three chiral superfields S_i . The action is given by

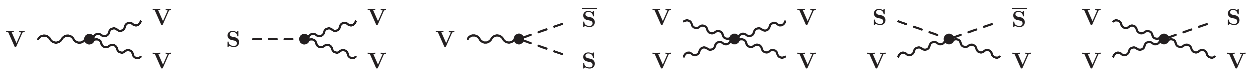
$$\begin{aligned} \mathcal{S}_{\text{10D}} = \frac{1}{g^2} \int d^{10}x \, \Big\{ \int d^2\theta \, \Big(\frac{1}{4} \, W^\alpha W_\alpha + \frac{1}{2} \, \epsilon^{IJK} S_I \partial_J S_K \Big) + \text{h.c.} + \\ + \int d^4\theta \, \Big[\Big(\sqrt{2} \, \bar{\partial}_I V - \overline{S}_I \Big) \Big(\sqrt{2} \, \partial^I V - S^I \Big) - \bar{\partial}_I V \partial^I V \Big] \Big\}. \end{aligned} \tag{2.8}$$

The non–Abelian extension of this action can be found in the literature [11, 12]. As that action by itself is not supergauge invariant, one needs to add an additional Wess–Zumino–Witten–like term [11]. However, for the applications discussed in the present work, this complication is irrelevant and therefore we do not give it explicitly here.

Supergraphs in extra dimensions

One of the central properties of supersymmetric theories are the many cancellations between bosonic and fermionic loops. In component computations such cancellations seem to be rather accidental. By using supergraphs [6–8], i.e. Feynman rules for superfields, these cancellations are built in from the very beginning. Supersymmetric theories in higher dimensions do not possess such simple and complete

Figure 2.3: These vertices describe the self interactions of the gauge multiplet.



off–shell formulations. Harmonic superspace [14,15] can be used in five/six dimensions [16]. However, this huge formalism hardly weighs up to the simplifications it brings to compute graphs in the end. Therefore, we take another approach: Obtain supergraphs for the four dimensional superfields that describe higher dimensional supersymmetric theories. The main advantage is that one can still use the familiar and simple four dimensional super Feynman rules without having to deal with harmonic superspace.

The supergraph methods in extra dimensions have been developed in [17–21]. Here we only sketch these methods in five dimensions. (Extensions to higher dimensions are straightforward except for some issues in ten dimensions which we comment on briefly below.) We determine the propagators of the superfields Φ_+, Φ_-, V and S by coupling them to the sources J_+, J_-, J_V and J_S, respectively. By considering the hyper multiplet action (2.6) and using some superspace identities, we obtain

$$\mathcal{S}_{\mathbb{H}2} = \int d^5x\, \mathrm{d}^4\theta \left(J_+ \overline{J}_-\right) \frac{-1}{\Box + \partial_5^2} \begin{pmatrix} 1 & \partial_5 \frac{D^2}{-4\Box} \\ -\partial_5 \frac{\overline{D}^2}{-4\Box} & 1 \end{pmatrix} \begin{pmatrix} \overline{J}_+ \\ J_- \end{pmatrix} . \tag{2.9}$$

As for standard massive chiral multiplets in four dimensions we have both non-chiral propagators between $\overline{J}_\pm$ and $J_\pm$, as well as chiral ones between J_+ and J_- and their conjugates. They are depicted in figure 2.1.

The vector multiplet requires more work because of gauge invariance. The problem of resulting zero modes can be made manifest by representing the quadratic action (2.7) in the following matrix form

$$\mathcal{S}_{\mathbb{V}2} = \frac{1}{2}\int d^5x\, d^4\theta\, \mathrm{tr} \begin{pmatrix} V & S & \overline{S} \end{pmatrix} A \begin{pmatrix} V \\ S \\ \overline{S} \end{pmatrix}, \qquad A = \begin{pmatrix} -\Box P_V - \partial_5^2 & \frac{1}{2}\sqrt{2}\, P_+ \partial_5 & \frac{1}{2}\sqrt{2}\, P_- \partial_5 \\ -\frac{1}{2}\sqrt{2}\, P_- \partial_5 & 0 & \frac{1}{2} P_- \\ -\frac{1}{2}\sqrt{2}\, P_+ \partial_5 & \frac{1}{2} P_+ & 0 \end{pmatrix}, \tag{2.10}$$

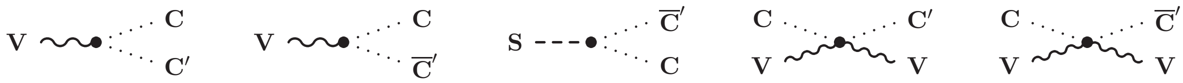
using the transversal projector $P_V = \frac{D^\alpha \overline{D}^2 D_\alpha}{-8\Box}$ and its chiral counterparts $P_+ = \frac{\bar{D}^2 D^2}{16\Box}$ and $P_- = \frac{D^2 \bar{D}^2}{16\Box}$. The zero modes of operator A correspond to the super gauge directions. The procedure to determine the gauge fixed action follows the conventional superfield methods for gauge multiplets. As usual we choose a gauge fixing functional

$$\Theta = \frac{\bar{D}^2}{-4}\left(\sqrt{2} V + \frac{1}{\Box}\partial_5 \bar{S}\right). \tag{2.11}$$

This gauge fixing functional has been previously considered in refs. [17,18]. The imaginary part of the restriction $\frac{D^2}{-4}\Theta| = \frac{1}{\sqrt{2}}\left(\Box C + D + \partial_5 \varphi - \mathrm{i}\partial_M A^M\right)$ reveals that the five dimensional Lorentz invariant gauge fixing, $\partial_M A^M = 0$, is incorporated. The gauge fixing action

$$\mathcal{S}_{\mathrm{gf}} = -\int d^5x\, d^4\theta\, \mathrm{tr}\left[\overline{\Theta}\Theta\right], \tag{2.12}$$

Figure 2.4: The ghosts C and C' only interact with the vector multiplet superfields V and S.



combined with (2.10) gives rise to invertible quadratic operators, and the mixing between the V and the S and $\overline{S}$ fields is removed. Consequently, the propagators, depicted in figure 2.1, for V and S decouple

$$\mathcal{S}_{\mathbb{V}2'} = \int d^5x\, d^4\theta\, \mathrm{tr} \left[\frac{1}{4} J_V \frac{1}{\Box + \partial_5^2} J_V + \overline{J}_S \frac{-1}{\Box + \partial_5^2} J_S \right] . \tag{2.13}$$

This decoupling amounts to a major simplification of the supergraph computations performed in later sections. The superfield S can be thought of as Goldstone boson superfield, therefore, the gauge fixing is an application of the supersymmetric 't Hooft R_ξ gauge [22].

To finish the description of the gauge fixing procedure, we need to introduce the ghost action

$$\begin{aligned} \mathcal{S}_{\mathrm{gh}} = \frac{1}{\sqrt{2}} \int d^5x\, d^4\theta\ \mathrm{tr} \Big[& \sqrt{2} \left(C' + \bar{C}' \right) \left(L_V \left(C - \bar{C} \right) + \coth\left(L_V \right) L_V \left(C + \bar{C} \right) \right) \\ & + C' \frac{\partial_5}{\Box} \left(\sqrt{2} \partial_5 \bar{C} - 2 \left[\bar{S}, \bar{C} \right] \right) + \bar{C}' \frac{\partial_5}{\Box} \left(\sqrt{2} \partial_5 C + 2 \left[S, C \right] \right) \Big] , \end{aligned} \tag{2.14}$$

where C and C' are anti–commuting chiral multiplets. From this action the ghost propagators can be read off

$$\mathcal{S}_{\mathrm{gh}2} = \int d^5x\, d^4\theta\, \mathrm{tr} \left[-\overline{J}'_C \frac{1}{\Box + \partial_5^2} J_C - J'_C \frac{1}{\Box + \partial_5^2} \overline{J}_C \right] , \tag{2.15}$$

and are given in figure 2.1.

As usual the interactions can be obtained by functional differentiation with respect to these sources, after the original superfields are integrated out using their corresponding quadratic actions. This gives rise to the super vertices displayed in figures 2.2, 2.3, 2.4. The extension to six dimensions is straightforward: One only needs to carefully replace the derivative ∂_5 by ∂ or $\bar{\partial}$, see e.g. [20]. In ten dimensions a few new features appear [11]: First of all in the quadratic action (2.8) there are superpotential terms proportional to the three dimensional complex epsilon tensor. This results in additional chiral propagators for the adjoint chiral superfields S_i. The second effect consequence is that there are now triple chiral interaction vertices involving these chiral adjoints.

2.2 Geometrical Description of Orbifolds

As the main topic of this work is to study the behavior of supersymmetric field theory and strings on orbifolds, we give a detailed introduction to orbifolds. We begin by giving a general overview orbifolds from a geometrical point of view. To define an orbifold we start with a covering space which possesses some finite number of discrete isometries, and define the orbifold as the covering space with these discrete isometries modded out.

To see how this works in practice we consider the simplest case $S^1/\mathbb{Z}_2$: Take the circle S^1 defined by the identification $y \to y + R$, with R its radius. Then the orbifold $S^1/\mathbb{Z}_2$ is obtained by modding out the reflection symmetry

$$y \;\rightarrow\; -y\;. \tag{2.16}$$

The resulting space is a line section including both end points, which are the fixed points 0 and $R/2$ of this discrete symmetry. The orbifold is not differentiable at these fixed points which means that curvature and other singularities could arise there.

Higher dimensional examples can be obtained by starting with flat torus T^{2n}, defined as a quotient of $\mathbb{C}^n$ divided by a lattice $\Lambda_W \sim \mathbb{Z}^{2n}$. Let θ be a $2n \times 2n$ matrix such that $\theta^p = \mathbb{1}$ and $\theta\Lambda_W = \Lambda_W$. Then we can defined the $T^{2n}/\mathbb{Z}_p$ orbifold by the action

$$z \;\rightarrow\; \Theta\, z\;, \qquad \Theta \;=\; e^{2\pi i\,\phi}\;, \qquad \phi \;=\; (\phi_1, \ldots, \phi_n)\;, \tag{2.17}$$

with $p\,\phi_i \equiv 0$, i.e. zero modulo integers. The orbifold action has to be crystallographicaly compatible with the lattice defining the torus. In this work we will be particular interested in the following six dimensional orbifolds:

$T^6/\mathbb{Z}_3$ Orbifold

The orbifold $T^6/\mathbb{Z}_3$ is defined on the torus

$$z_i \;\sim\; z_i \;+\; R_i \;\sim\; z_i \;+\; R_i\,\theta\;, \tag{2.18}$$

where we have introduced the third root of unity $\theta = e^{2\pi i/3}$. On this torus we can define the following $\mathbb{Z}_3$ action

$$z_i \;\rightarrow\; \theta\, z_i\;. \tag{2.19}$$

The 27 fixed points of this orbifold can be parameterized by $\mathfrak{z}_s = (R_1\zeta_{s_1}, R_2\zeta_{s_2}, R_3\zeta_{s_3})$, for $s_1, s_2, s_3 = 0, 1, 2$ with $\zeta_0 = 0$, $\zeta_1 = (1+2\theta)/3$, and $\zeta_1 = (2+\theta)/3$.

$T^6/\mathbb{Z}_4$ Orbifold

In the previous two examples of all fixed points of the orbifold were equivalent in the sense, that they are all fixed points of the full orbifold group. The reason that all the fixed points are equivalent in these cases is that the orbifold group does not have proper subgroups, because they were primed orbifolds. An example of an orbifold in which not all fixed points are equivalent is given by $T^6/\mathbb{Z}_4$, see e.g. [23–26]. Consider the square torus

$$z_j \;\sim\; z_j \;+\; R_j \;\sim\; z_j \;+\; i\,R_j\;, \tag{2.20}$$

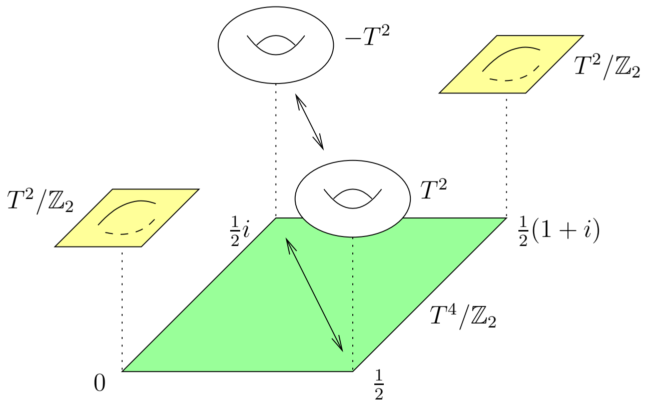
on which we can define the $\mathbb{Z}_4$ orbifold action

$$(z_1, z_2, z_3) \;\rightarrow\; (-z_1, i\,z_2, i\,z_3)\;. \tag{2.21}$$

To describe the fixed points we introduce the following notation: $\zeta_{s\,t} = (s + i\,t)/2$ for $s, t \in \{0, 1\}$. The fixed points of the twists Θ and Θ^3 are the 16 different $\mathbb{Z}_4$ fixed points:

$$\mathfrak{z}^4_{s\,t} \;=\; (R_1\zeta_{s_1t_1}, R_2\zeta_{s_2s_2}, R_3\zeta_{s_3s_3})\;. \tag{2.22}$$

Figure 2.5: An impression of the two dimensional fixed hyper surfaces within the orbifold $T^6/\mathbb{Z}_4$ is given. The bottom square represents a part of a two dimensional cross section (the z_2 plane) of the orbifold $T^4/\mathbb{Z}_4$. Fixed orbifolds $T^2/\mathbb{Z}_2$ are located above its fixed points 0 and $\frac{1}{2}(1+i)$. Because the points $\frac{1}{2}$ and $\frac{1}{2}i$ are mapped to each other the two–tori T^2 above them are identified.



In addition to these fixed points of the $\mathbb{Z}_4$ action, there are two dimensional surfaces within $T^6/\mathbb{Z}_4$: The Θ^2 fixed hyper surfaces take the form of 16 disjoint two tori

$$(T^2, \mathfrak{z}^2_{s\,t}) \;=\; (T^2, R_2\zeta_{s_2t_2}, R_3\zeta_{s_3t_3})\;, \tag{2.23}$$

in the covering torus of the orbifold. Since on the fixed space of Θ^2 the $\mathbb{Z}_4$ twist Θ acts non–trivially

$$\Theta\,(z_1, \mathfrak{z}^2_{s\,t}) \;=\; (-z_1, \mathfrak{z}^2_{t\,s}) - (0, t_2R_2, t_3R_3), \tag{2.24}$$

the embedding of this fixed space in the orbifold $T^6/\mathbb{Z}_4$ is more involved. Because this action interchanges the order of p and q, it is important to distinguish between the $\mathbb{Z}_2$ fixed points with p and q equal or not: The twist Θ leaves $\mathfrak{z}^2_{s=t}$ invariant, and therefore creates four $T^2/\mathbb{Z}_2$ orbifolds

$$(T^2/\mathbb{Z}_2, \mathfrak{z}^2_{s=t})\;, \qquad \mathfrak{z}^2_{s=t} \;=\; (R_2\zeta_{s_2s_2}, R_3\zeta_{s_3s_3})\;. \tag{2.25}$$

As each orbifold $T^2/\mathbb{Z}_2$ has four fixed points $R_1\zeta_{s_1t_1}$, the number of fixed points of all four disjunct orbifolds together is precisely the same as all fixed points $\mathfrak{z}^4_{s\,t}$ of the original orbifold $T^6/\mathbb{Z}_4$. On the other 12 two–tori the twist Θ acts freely; this leads to an identification of the pairs of two–tori $(T^2, \mathfrak{z}^2_{s\neq t})$ and $(-T^2, \mathfrak{z}^2_{s\neq t})$ in the covering space of the orbifold $T^6/\mathbb{Z}_4$: The orbifold $T^6/\mathbb{Z}_4$ only contains six fixed two–tori. A schematic picture of the embedding of the fixed two–tori T^2 and orbifolds $T^2/\mathbb{Z}_2$ within $T^6/\mathbb{Z}_4$ is given in figure 2.5.

2.3 Fields on Orbifolds

After having discussed various supersymmetric theories in extra dimensions and introduced orbifolds, we are ready to describe field theories on orbifolds. In this work we are mostly interested in supersymmetric theories, therefore we employ the superfield language. For simple orbifolds, like $S^1/\mathbb{Z}_2$, the Kaluza–Klein expansion approach has been widely used to describe orbifold field theories (see e.g. [27–32]). This method deals whole Kaluza–Klein towers of infinitely many four dimensional states. For $\mathbb{Z}_2$ orbifolds dealing with such towers is manageable (even though a lot of confusion arose in the literature about their precise properties), they become extremely inconvenient for more complicated orbifolds We present an approach that does not require any Kaluza–Klein expansions, by introducing orbifold compatible fields and delta functions [19, 21, 33].

We consider the generic case of a chiral multiplet on $T^{2n}/\mathbb{Z}_p$. This chiral multiplet Φ is periodic function in the various directions of T^{2n}

$$\Phi(z + 2\pi R_i) = T_i \Phi(z) , \tag{2.26}$$

possibly upto transformations T_i, which constitute symmetries of its action. In addition it transforms under the orbifold symmetry

$$\Phi(\Theta z) = U \Phi(z) , \tag{2.27}$$

where U should also be a symmetry of the action and have the same order as the orbifold group $\mathbb{Z}_p$, i.e. $U^p = \mathbb{1}$. These two actions have to be compatible, which requires that $(TU)^p = \mathbb{1}$. This means that both U and T are quantized. Sometimes such a T is called a discrete Wilson line, because by a field redefinition T can be reformulated as a constant gauge background, and the superfield Φ becomes strictly periodic on T^{2n}. In this work we assume throughout that the orbifold twist and the Wilson lines commute, this means that we can choose a Cartan subgroup in which they both lie. This Cartan subgroup is generated by H_I. The remaining part of the gauge group is then generated by algebra elements E_w, where w are the weights of the adjoint representation of the gauge group. We can then represent U and T_i as

$$U = e^{2\pi i\, v^I H_I} , \qquad T_i = e^{2\pi i\, a_i^I H_I} . \tag{2.28}$$

In particular in order that this defines $\mathbb{Z}_p$ actions on the adjoint of E_8 and the adjoint and spinorial representation of SO(32) we require that pv and pa_i are either vectors with either all integer or all half–integer entries.

Assuming that we work in the basis where Φ is strictly periodic, we can construct an orbifold compatible superfield [19, 21, 33], satisfying (2.27), from a generic function Φ on the torus T^{2n}

$$\tilde{\Phi}(y) = \frac{1}{p} \sum_{n=0}^{p-1} U^n \, \Phi(\Theta^{-n} z) . \tag{2.29}$$

To derive Feynman rules for (super)graphs one needs to define functional differentiation, therefore, it is convenient to define orbifold compatible delta functions. Because the source $\widetilde{J}$ for the superfield $\tilde{\Phi}$ has precisely the opposite orbifold boundary conditions, the orbifold compatible delta function reads

$$\widetilde{\delta}_{21} = \frac{1}{p} \sum_{n=0}^{p-1} U^n \delta(z_2 - \Theta^{-n} z_1) , \tag{2.30}$$

where the flat four dimensional $\delta^4(x_2 - x_1)$ and the chiral superspace delta function $(\theta_2 - \theta_1)^2$ are kept implicit. Using these orbifold compatible fields it is not difficult to derive a general formula [34] for the field theoretical Hilbert space trace, denoted by Tr, of a generic operator $\mathcal{O}$. We will not give the general formula here, but only give its representations for the orbifolds considered above.

On $S^1/\mathbb{Z}_2$ with Wilson line T and orbifold action U on the bundle, the one–loop trace of some operator $\mathcal{O}(x^m, y; \partial_m, \partial_y)$ that acts on this bundle reads

$$\mathrm{Tr}_{S^1/\mathbb{Z}_2,T,U}\Big[\mathcal{O}\Big] \;=\; \frac{1}{2}\,\mathrm{Tr}_{S^1/\mathbb{Z}_2,T}\Big[\mathcal{O}\Big] \;+\; \frac{1}{2}\cdot\frac{1}{2}\sum_{s=0,1}\mathrm{Tr}_{\mathfrak{z}_s}\Big[U_s\mathcal{O}'\Big]\;. \tag{2.31}$$

The first turn in the expression corresponds to the bulk contribution; the fixed points are merely treated as open end points of the interval. When computing this bulk trace one should nevertheless keep track of the effect of the Wilson line, i.e. the periodicity upto T in the bundle. The second term gives the sum of the contributions from both fixed point $\mathfrak{z}_0 = 0$ and $\mathfrak{z}_1 = R/2$. Because of the appearance of $U_s = UT^s$ the trace at these two fixed points is in general not the same when a Wilson line T is present. The multiplicity factors in (2.31) can be understood as follows: As this orbifold trace is derived using orbifold compatible fields (2.29), we get two different types of contributions normalized with a factor $1/2$. Moreover, at the fixed points we encounter $\delta(2y)$ on $S^!$, see (2.30), which can be expanded as a sum over the fixed points

$$\delta(2y) \;=\; \frac{1}{2}\delta(y) \;+\; \frac{1}{2}\delta(y - R/2)\;. \tag{2.32}$$

Hence the second factor $1/2$ in the second term of (2.31) should be understood as one over the number of fixed points. Because this requires a rescaling of the y coordinate, the operator $\mathcal{O}$ on the fixed points has to be modified to $\mathcal{O}' = \mathcal{O}(x^m, y; \partial_m, \frac{1}{2}\,\partial_y)$.

On more complicated orbifolds similar trace formulae can be derived. Here we content ourselves with giving the generalizations for the orbifolds $T^6/\mathbb{Z}_3$ and $T^6/\mathbb{Z}_4$. The trace formula on the orbifold $T^6/\mathbb{Z}_3$ takes the form

$$\mathrm{Tr}_{T^6/\mathbb{Z}_3,T,U}\Big[\mathcal{O}\Big] \;=\; \frac{1}{3}\,\mathrm{Tr}_{T^6/\mathbb{Z}_3,T}\Big[\mathcal{O}\Big] \;+\; \frac{1}{3}\cdot\frac{1}{27}\sum_{s}\mathrm{Tr}_{\mathfrak{z}_s}\Big[U_s\,\mathcal{O}_1 \;+\; U_s^2\,\mathcal{O}_2\Big]\;, \tag{2.33}$$

where now $\mathcal{O}_n = \mathcal{O}\big(x^m, z_i; \partial_m, (1-\theta^n)^{-1}\partial_i\big)$ and U_s is defined in terms of the local gauge shift vector

$$U_s \;=\; e^{2\pi i\, v_s\, H_I}\;, \qquad v_s \;=\; v \;+\; s_i\, a_i\;. \tag{2.34}$$

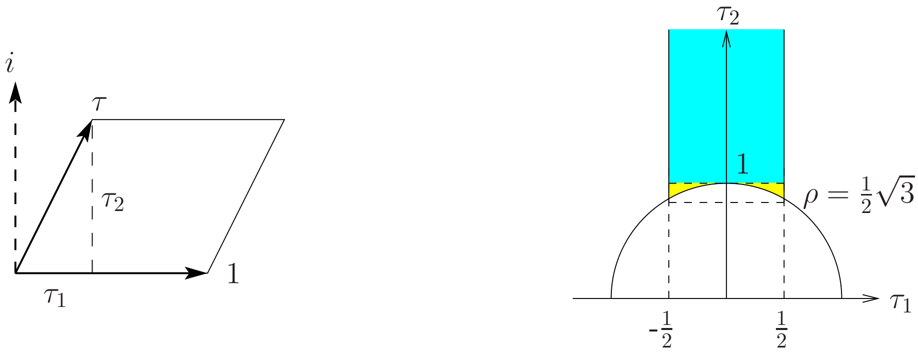
The $T^6/\mathbb{Z}_4$ orbifold trace is more complicated, because we have to take into account that the orbifold does not only have fixed points, but also fixed tori T^2 and orbifolds $T^2/\mathbb{Z}_2$, see figure 2.5. The orbifold trace can be written as

$$\begin{aligned}\mathrm{Tr}_{T^6/\mathbb{Z}_4,T,U}\Big[\mathcal{O}\Big] \;&=\; \frac{1}{4}\,\mathrm{Tr}_{T^6/\mathbb{Z}_4}\Big[\mathcal{O}\Big] \;+\; \frac{1}{4}\cdot\frac{1}{16}\sum_{s,t}\mathrm{Tr}_{(\mathbb{R}^4,\mathfrak{z}^4_{s,t})}\Big[U^4_{s,t}\mathcal{O}_1 + (U^4_{s,t})^{-1}\mathcal{O}_{-1}\Big] +\\ &+\; \frac{1}{4}\cdot\frac{1}{16}\sum_{s\neq t}\mathrm{Tr}_{(T^2,\mathfrak{z}^2_{s\neq t})}\Big[U^2_{s,t}\mathcal{O}_2\Big] \;+\; \frac{1}{4}\cdot\frac{1}{16}\sum_{s=t}\mathrm{Tr}_{(T^2/\mathbb{Z}_2,\mathfrak{z}^2_{s=t})}\Big[U^2_{p,q}\mathcal{O}_2\Big],\end{aligned} \tag{2.35}$$

where $\mathcal{O}_{\pm 1} \;=\; \mathcal{O}\Big(x, z; \frac{1}{2}\partial_1, \frac{1\pm i}{2}\partial_2, \frac{1\pm i}{2}\partial_3\Big)$ and $\mathcal{O}_2 \;=\; \mathcal{O}\Big(x, z; \partial_1, \frac{1}{2}\partial_2, \frac{1}{2}\partial_3\Big)$,and the matrices

$$\begin{aligned} U^4_{s,t} \;&=\; e^{2\pi i\, v^4_{s,t}\, H_I}\;, \qquad v^4_{s\,t} \;=\; s_1\, a_1 + t_1\, a_1' \;+\; s_2\, a_2 \;+\; s_3\, a_3 + v\;,\\ U^2_{s,t} \;&=\; e^{2\pi i\, v^2_{s,t}\, H_I}\;, \qquad v^2_{s\,t} \;=\; (s_2 + t_2)\, a_2 \;+\; (s_3 + t_3) a_3 + 2\,v\;. \end{aligned} \tag{2.36}$$

Figure 2.6: The first picture gives a torus defined by the complex variable τ. To only label inequivalent tori this parameter is restricted to the fundamental domain $\mathcal{F}$, depicted in the second picture.



world sheet torus

$$\mathcal{F} = \left\{ \tau \in \mathbb{C} \,|\, \text{-}\tfrac{1}{2} < \tau_1 < \tfrac{1}{2},\ \tau_2 > \sqrt{1-\tau_1^2} \right\},$$

are given in terms of the local shift vectors of the different $\mathbb{Z}_4$ and $\mathbb{Z}_2$ fixed points and surfaces. The first term of (2.35) is the bulk contribution. The second term gives the contributions at the four dimensional fixed points of $T^6/\mathbb{Z}_4$. The terms on the second line encode the effects at the fixed T^2's and$T^2/\mathbb{Z}_2$.

2.4 Heterotic Strings on Orbifolds

Green and Schwarz showed that it is possible to construct anomaly free ten dimensional supergravity coupled to super Yang–Mills theory [35,36] provided that the gauge group is either SO(32) or $\mathrm{E}_8 \times \mathrm{E}_8$. The heterotic string has been introduced by [37, 38] to provide explicit string constructions, that have these supergravity theories as their low energy limit. The name heterotic refers to fact that the construction treats the left and right movers of the string very differently.

We give a more technical description of the heterotic string from the world sheet point of view, which we need to present our results section 4.2. The world sheet is parameterized by the complex coordinate σ . On this world sheet conformal field theories live: $X^M(\sigma)$ are the target space coordinate fields and $\psi^M(\sigma)$ their right–moving fermionic partners. The part of the theory encoding the gauge structure is described by the left–moving fermions $\lambda_a^{2I}(\sigma), \lambda_a^{2I+1}(\sigma)$. For the $\mathrm{E}_8 \times \mathrm{E}_8{}'$ theory there are two sets of fermions, labeled by $a = 1, 2$, of $I = 1, \ldots 8$. The SO(32) string contains $I = 1, \ldots 16$ fermions and $a = 1$. The full partition function of the heterotic string on an orbifold is constructed by summing over all partition functions corresponding to different world sheet boundary conditions compatible with the orbifold. The terms in this sum can be multiplied by non–trivial phases. These phases are restricted by the requirement of modular invariance of the one loop string amplitude [39–45]: The Teichmüller parameter τ defines the world sheet torus periodicities: $\sigma \sim \sigma + 1$ and $\sigma \sim \sigma + \tau$. However not all choices of τ lead to inequivalent tori. In particular the modular transformations $\tau \to \tau + 1$ and $\tau \to -1/\tau$ relate equivalent tori. In figure 2.6 we have depicted the fundamental domain $\mathcal{F}$ which is defined by $|\tau| \geq 1$ and $-\frac{1}{2} \leq \tau_1 \leq \frac{1}{2}$. The requirement of modular invariance gives a rather strong restriction on the possible phases, implements various GSO and orbifold projections,

Table 2.1: This table gives the complete spectrum of the SO(32) heterotic $\mathbb{Z}_3$ orbifold models with the gauge shift vector $v = \frac{1}{3}(0^{16-3n}, 1^{2n}, \text{-}2^n)$.

n	gauge group	untwisted (x 3)	twisted (x 27)
0	SO(32)		$9(\mathbf{1})$
1	SO(26) × SU(3) × U(1)	$(\mathbf{26},\mathbf{3})_{\mathbf{1}} + (\mathbf{1},\mathbf{3})_{\text{-}2}$	$3(\mathbf{1},\overline{\mathbf{3}})_{\mathbf{0}} + (\mathbf{1},\mathbf{1})_{\text{-}2} + (\mathbf{26},\mathbf{1})_1$
2	SO(20) × SU(6) × U(1)	$(\mathbf{20},\mathbf{6})_1 + (\mathbf{1},\overline{\mathbf{15}})_{\text{-}2}$	$3(\mathbf{1},\mathbf{1})_2 + (\mathbf{1},\overline{\mathbf{15}})_0$
3	SO(14) × SU(9) × U(1)	$(\mathbf{14},\mathbf{9})_1 + (\mathbf{1},\overline{\mathbf{36}})_{\text{-}2}$	$(\mathbf{1},\overline{\mathbf{9}})_2$
4	SO(8) × SU(12) × U(1)	$(\mathbf{8},\mathbf{12})_1 + (\mathbf{1},\overline{\mathbf{66}})_{\text{-}2}$	$(\mathbf{1},\mathbf{1})_{\mathbf{4}} + (\mathbf{8}_+,\mathbf{1})_{\text{-}2}$
5	SU(15) × U(1) × U(1)′	$(\mathbf{15})_{1,\text{-}1} + (\mathbf{15})_{1,1} + (\overline{\mathbf{105}})_{\text{-}2,0}$	$3(\mathbf{1})_{\text{-}\frac{\mathbf{5}}{\mathbf{2}},\text{-}\frac{\mathbf{1}}{\mathbf{2}}} + (\mathbf{15})_{\text{-}\frac{3}{2},\frac{1}{2}}$

and determines the twisted sectors. After the string partition function has been determined, one can extract the massless spectrum. This gives ten dimensional SO(32) or $\text{E}_8 \times \text{E}_8{}'$ theory coupled to supergravity, and the twisted spectrum at the orbifold fixed points.

String on orbifolds have been first considered in [46–50]. The heterotic strings on orbifolds needs to fulfill all conditions their low energy effective field theories satisfy, but there are some additional conditions. First of all, the geometrical shift vector ϕ, the gauge shift vector v and Wilson lines a_i need to have proper $\mathbb{Z}_p$ actions on spinors, this requires that

$$\sum_i \phi_i \;=\; 0 \bmod 2 \;, \qquad \sum_I v^I \;=\; 0 \bmod 2 \;, \qquad \sum_I a_i^I \;=\; 0 \bmod 2 \;. \tag{2.37}$$

In addition, these shifts and Wilson lines have to satisfy all possible local modular invariance conditions

$$v_s^2 \;-\; \phi^2 \;=\; 0 \bmod 2 \;, \tag{2.38}$$

where the local gauge shifts v_s are defined in (2.34). Together these conditions are very restrictive and imply that only a limited number of heterotic $\mathbb{Z}_3$ and $\mathbb{Z}_4$ models exists without Wilson lines. (In models with Wilson lines the local spectra at the various fixed points are determined by the local gauge shift vectors v_s.) Moreover as was emphasized in [51] the twisted spectra of such heterotic orbifolds fall into regular patterns of representations. Tables 2.1 and 2.3 give the massless spectra of $T^6/\mathbb{Z}_3$ orbifold models of the heterotic SO(32) and $\text{E}_8 \times \text{E}_8{}'$ strings, respectively. Table 2.2 gives the chiral part of the spectra of $\mathbb{Z}_4$ SO(32) orbifolds which clear exhibits such regular patterns.

Table 2.2: The six and four dimensional gauge groups are tabulated of the SO(32) heterotic $\mathbb{Z}_4$ orbifold models defined by the gauge shift vector $v = \frac{1}{4}\big(0^{n_0}, 1^{n_1}, 2^{n_2}\big)$, $n_0 = 16 - n_1 - n_2$. The six dimensional (half) hyper multiplets included the multiplicity factors that count the number of independent $T^4/\mathbb{Z}_2$ fixed points within $T^6/\mathbb{Z}_4$. The four dimensional twisted states and zero modes of the twisted states on $T^2/\mathbb{Z}_2$ complete the table. This table does not give the complete four dimensional spectrum, only the chiral part, relevant for anomaly considerations.

n_1	6D gauge group	6D untwisted $\mathbf{R}$	6D twisted $\mathbf{D}, \mathbf{S}$	
n_2	4D gauge group	4D untwisted $\mathbf{R}_{i=1,2}$	6D twisted on $T^2/\mathbb{Z}_2$	4D twisted $\mathbf{T}$
2	SO(28) × SO(4)	$(\mathbf{28},\mathbf{4}) + 4(\mathbf{1},\mathbf{1})$	$20(\mathbf{1},\mathbf{2}_-) + 5(\mathbf{28},\mathbf{2}_+)$	
1	SO(26) × U(2) × U(1)′	$(\mathbf{26},\mathbf{2})_{1,0} + (\mathbf{1},\overline{\mathbf{2}})_{-1,\pm1}$	$2(\mathbf{1},\mathbf{2})_{0,0} + (\mathbf{26},\mathbf{1})_{1,0} + (\mathbf{1},\mathbf{1})_{-1,\pm1}$	$5(\mathbf{1},\mathbf{1})_{\frac{1}{2},\frac{1}{2}} + 2(\mathbf{1},\overline{\mathbf{2}})_{-\frac{1}{2},-\frac{1}{2}}$ $+(\mathbf{1},\mathbf{1})_{-\frac{3}{2},\frac{1}{2}} + (\mathbf{26},\mathbf{1})_{\frac{1}{2},-\frac{1}{2}}$
3	SO(22) × U(2) × SO(6)′	$(\mathbf{22},\mathbf{2},\mathbf{1})_1 + (\mathbf{1},\overline{\mathbf{2}},\mathbf{6})_{-1}$	$2(\mathbf{1},\mathbf{2},\mathbf{1})_0 + (\mathbf{22},\mathbf{1},\mathbf{1})_{-1} + (\mathbf{1},\mathbf{1},\mathbf{6})_1$	$2(\mathbf{1},\mathbf{1},\mathbf{4}_+)_{\frac{1}{2}} + (1,\mathbf{2},\mathbf{4}_-)_{-\frac{1}{2}}$
5	SO(18) × U(2) × SO(10)′	$(\mathbf{18},\mathbf{2},\mathbf{1})_1 + (\mathbf{1},\overline{\mathbf{2}},\mathbf{10})_{-1}$	$2(\mathbf{1},\mathbf{2},\mathbf{1})_0 + (\mathbf{18},\mathbf{1},\mathbf{1})_1 + (\mathbf{1},\mathbf{1},\mathbf{10})_{-1}$	$(\mathbf{1},\mathbf{1},\mathbf{16}_+)_{\frac{1}{2}}$
7	SO(14) × U(2) × SO(14)′	$(\mathbf{14},\mathbf{2},\mathbf{1})_1 + (\mathbf{1},\overline{\mathbf{2}},\mathbf{14})_{-1}$	$2(\mathbf{1},\mathbf{2},\mathbf{1})_0 + (\mathbf{14},\mathbf{1},\mathbf{1})_{-1} + (\mathbf{1},\mathbf{1},\mathbf{14})_1$	
6	SO(20) × SO(12)	$(\mathbf{20},\mathbf{12}) + 4(\mathbf{1},\mathbf{1})$	$5(\mathbf{1},\mathbf{32}_+)$	
0	SO(20) × U(6)	$(\mathbf{20},\mathbf{6})_1$	$(\mathbf{1},\mathbf{1})_{-3} + (\mathbf{1},\overline{\mathbf{15}})_1$	$5(\mathbf{1},\mathbf{1})_{\frac{3}{2}} + (\mathbf{1},\overline{\mathbf{15}})_{-\frac{1}{2}}$
2	SO(16) × U(6) × SO(4)′	$(\mathbf{16},\mathbf{6},\mathbf{1})_1 + (\mathbf{1},\overline{\mathbf{6}},\mathbf{4})_{-1}$	$(\mathbf{1},\mathbf{1},\mathbf{1})_3 + (\mathbf{1},\mathbf{15},\mathbf{1})_{-1}$	$2(\mathbf{1},\mathbf{1},\mathbf{2}_+)_{\frac{3}{2}} + (\mathbf{1},\overline{\mathbf{6}},\mathbf{2}_-)_{\frac{1}{2}}$
4	SO(12) × U(6) × SO(8)′	$(\mathbf{12},\mathbf{6},\mathbf{1})_1 + (\mathbf{1},\overline{\mathbf{6}},\mathbf{8}_v)_{-1}$	$(\mathbf{1},\mathbf{1},\mathbf{1})_{-3} + (\mathbf{1},\overline{\mathbf{15}},\mathbf{1})_1$	$(\mathbf{1},\mathbf{1},\mathbf{8}_+)_{\frac{3}{2}}$
10	SO(12) × SO(20)	$(\mathbf{12},\mathbf{20}) + 4(\mathbf{1},\mathbf{1})$	$5(\mathbf{32}_-,\mathbf{1})$	
1	SO(10) × U(10) × U(1)′	$(\mathbf{10},\mathbf{10})_{1,0} + (\mathbf{1},\overline{\mathbf{10}})_{-1,\pm1}$	$(\mathbf{16}_-,\mathbf{1})_{0,-\frac{1}{2}}$	$2(\mathbf{1},\mathbf{1})_{\frac{5}{2},\frac{1}{2}} + (\mathbf{1},\overline{\mathbf{10}})_{\frac{3}{2},-\frac{1}{2}}$
3	SO(6) × U(10) × SO(6)′	$(\mathbf{6},\mathbf{10},\mathbf{1})_1 + (\mathbf{1},\overline{\mathbf{10}},\mathbf{6})_{-1}$	$(\mathbf{4}_-,\mathbf{1},\mathbf{4}_+)_0$	$(\mathbf{1},\mathbf{1},\mathbf{4}_+)_{\frac{5}{2}} + (\mathbf{4}_-,\mathbf{1},\mathbf{1})_{-\frac{5}{2}}$
14	SO(4) × SO(28)	$(\mathbf{4},\mathbf{28})$	$20(\mathbf{2}_+,\mathbf{1}) + 5(\mathbf{2}_-,\mathbf{28})$	
0	SO(4) × U(14)	$(\mathbf{4},\mathbf{14})_1$	$2(\mathbf{2}_+,\mathbf{1})_0 + (\mathbf{2}_-,\overline{\mathbf{14}})_{-1}$	$2(\mathbf{1},\mathbf{1})_{\frac{7}{2}} + (\mathbf{2}_+,\mathbf{1})_{-\frac{7}{2}}$

Table 2.3: The fixed point spectra of five of the eight $E_8 \times E_8'$ heterotic $\mathbb{Z}_3$ orbifold models are displayed.

model	gauge shift v and gauge group $G = G_s$	untwisted (x 3) $\mathbf{R} = \mathbf{R}_s$	Twisted $\mathbf{S}_s$	(x 27) $3 \times \mathbf{T}_s$
E_8	$\frac{1}{3}\big(0^8 \mid 0^8 \big)$ $E_8 \times E_8'$			$(\mathbf{1})(\mathbf{1})'$
E_6	$\frac{1}{3}\big(\text{-}2,\ 1^2,\ 0^5 \mid 0^8 \big)$ $E_6 \times SU(3) \times E_8'$	$(\mathbf{27},\overline{\mathbf{3}})(\mathbf{1})'$	$(\mathbf{27},\mathbf{1})(\mathbf{1})'$	$(\mathbf{1},\mathbf{3})(1)'$
${E_6}^2$	$\frac{1}{3}\big(\text{-}2,\ 1^2,\ 0^5 \mid \text{-}2,\ 1^2,\ 0^5\big)$ $E_6 \times SU(3) \times E_6' \times SU(3)'$	$(\mathbf{27},\overline{\mathbf{3}})(\mathbf{1},\mathbf{1})+(\mathbf{1},\mathbf{1})(\mathbf{27},\overline{\mathbf{3}})'$	$(\mathbf{1},\mathbf{3})(\mathbf{1},\mathbf{3})'$	
E_7	$\frac{1}{3}\big(0,\ 1^2,\ 0^5 \mid \text{-}2,\ 0^7 \big)$ $E_7 \times U(1) \times SO(14)' \times U(1)'$	$(\mathbf{1})_0(\mathbf{64})'_{\frac{1}{2}} + (\mathbf{56})_1(\mathbf{1})'_0$ $+ (\mathbf{1})_0(\mathbf{14})'_{\text{-}1} + (\mathbf{1})_{\text{-}2}(\mathbf{1})'_0$	$(\mathbf{1})_{\frac{2}{3}}(\mathbf{14})'_{\text{-}\frac{1}{3}}$ $+ (\mathbf{1})_{\text{-}\frac{4}{3}}(\mathbf{1})'_{\frac{2}{3}}$	$(\mathbf{1})_{\frac{2}{3}}(\mathbf{1})'_{\frac{2}{3}}$
SU(9)	$\frac{1}{3}\big(\text{-}2,\ 1^4\ ,0^3 \mid \text{-}2,\ 0^7 \big)$ $SU(9) \times SO(14)' \times U(1)'$	$(\mathbf{84})(\mathbf{1})'_0 + (\mathbf{1})(\mathbf{64})'_{\frac{1}{2}}$ $+ (\mathbf{1})(\mathbf{14})'_{\text{-}1}$	$(\overline{\mathbf{9}})(\mathbf{1})'_{\frac{2}{3}}$	

2.5 Conclusions

In summary we have developed the necessary materials for the discussion of anomalies, tadpoles, gauge corrections on and resolutions of orbifold singularities. Even though this section therefore gave an introduction to the subject, various novel techniques were described to perform explicit calculations on orbifolds using orbifold compatible field theory and orbifold traces. We will make extensive use of these techniques in later sections. In addition it gave an overview of heterotic SO(32) and $E_8 \times E_8$ strings on the orbifolds $T^6/\mathbb{Z}_3$ and $T^6/\mathbb{Z}_4$, which will be revisited in various places in this work.

Chapter 3

Consistency: Anomalies

As described in the introduction anomalies signal breakdown of symmetries at the quantum level. This is disastrous for gauge symmetries because unitarity is lost. There has been a lot of activity to understand and determine anomalies; we quote these well–known results for anomalies in the first subsection. After that we determine the structure of anomalies on orbifolds. With this technology we are then able to investigate local anomaly cancellation in heterotic orbifold compactifications.

3.1 Anomalies in Field Theories

We consider a left–handed chiral fermion χ in $D = 2n$ dimensions. We take into account the effects of a non–trivial background given by a gauge field A_M and a spin connection $\omega(e)$ as a function of the vielbein e in the Dirac operator. The Dirac operator of the fermion maps the Hilbert space of positive chirality to that of negative chirality. By introducing a non–interacting right–handed fermion ξ, a Dirac operator $\not{D}$ can be obtained for the Dirac spinor $\Psi = \chi + \xi$, that maps the total Hilbert space to itself. The action is classically gauge invariant

$$S[\Psi, A, e] = -\int \mathrm{d}^{2n}x\, e^{-1}\, \frac{1}{2}\, \overline{\Psi}\not{D}\Psi \quad \text{with} \quad \not{D} = \not{\partial} + (i\not{A} + \not{\omega})\frac{1+\widetilde{\Gamma}}{2}, \quad S[\,{}^g\Psi, {}^gA, e] = S[\Psi, A, e], \tag{3.1}$$

where $\widetilde{\Gamma}$ is the chirality operator. We have an anomaly if the effective path integral $Z[A]$, obtained by integrating out the fermions

$$Z[A] = \int \mathcal{D}\overline{\Psi}\mathcal{D}\Psi e^{iS[\Psi,A,e]} \neq Z[\,{}^gA] = Z[A]e^{i\mathcal{A}[g]}, \qquad \mathcal{A}[\Lambda] = \mathrm{Tr}[\Lambda\widetilde{\Gamma}], \tag{3.2}$$

is not gauge invariant: $\mathcal{A}[g] \neq 0$. In the last equation we have restricted ourselves to infinitesimal gauge transformations, denoted by Λ.

The trace Tr in (3.2) is formal, as it is taken over both spinor and gauge representations, as well as (over)countable number of states, and therefore requires regularization. To evaluate the formal trace, one can employ Fujikawa's method [52, 53] to regularize this trace by using the heat–kernel

$$\mathcal{A}[\Lambda] = \mathrm{Tr}\Big[\Lambda\widetilde{\Gamma}e^{-(\not{D}/M)^2}\Big] = \int \mathrm{d}^{2n}x\, \mathrm{tr}\Big[\Lambda\widetilde{\Gamma}e^{-(\not{D}/M)^2}\delta(x-x')\Big], \tag{3.3}$$

where the limits $x' \to x$, and $M \to \infty$ are to be taken once the anomaly expression is well–defined.

The structure of anomalies is not free; it has to satisfy the Wess-Zumino consistency condition [54]

$$\delta_{\Lambda_1}\mathcal{A}[\Lambda_2]-\delta_{\Lambda_2}\mathcal{A}[\Lambda_1]=\mathcal{A}[\Lambda_3], \qquad \Lambda_3=[\Lambda_1,\Lambda_2]. \tag{3.4}$$

This fixes the anomaly upto an overall factor: The form of anomaly is given as the integral over Ω^1_{2n}, which is determined from invariant anomaly polynomial Ω_{2n+2} by the descent equations [55]:

$$\Omega_{2n+2}=\mathrm{d}\Omega_{2n+1}=\mathrm{ch}(iF)\hat{A}(R), \quad \delta_\Lambda\Omega_{2n+2}=0, \quad \delta_\Lambda\Omega_{2n+1}=\mathrm{d}\Omega^1_{2n}(\Lambda). \tag{3.5}$$

Here ch is the Chern character and $\hat{A}$ is the roof genus of the curvature tensor R. Their expressions can be found in [55–58], for example.

We close this subsection by quoting the expressions for the anomaly polynomials in six and four dimensions. Let $\mathbf{r}$ be some representation of a (non–)Abelian gauge group. The six dimensional anomaly polynomial is given by

$$\Omega_{8|r}=\frac{-i}{(2\pi)^3}\left[\frac{1}{24}\mathrm{tr}_rF^4-\frac{1}{96}\mathrm{tr}_rF^2\mathrm{tr}R^2+\frac{\dim r}{128}\left(\frac{1}{45}\mathrm{tr}R^4+\frac{1}{36}(\mathrm{tr}R^2)^2\right)\right]. \tag{3.6}$$

For the four dimensional anomaly polynomial we have instead

$$\Omega_{6|\mathbf{r}}=\frac{-i}{(2\pi)^2}\Big[\frac{1}{3!}\mathrm{tr}_{\mathbf{r}}(iF)^3-\frac{1}{48}\mathrm{tr}R^2\mathrm{tr}_{\mathbf{r}}(iF)\Big]. \tag{3.7}$$

In this work we never need the expressions for the ten dimensional anomalies, therefore, we refrain from giving them here.

3.2 Anomalies on Orbifolds

To evaluate anomalies on orbifolds the simplest approach is to expand in eigenfunctions in the internal directions. For simple orbifolds, like $S^1/\mathbb{Z}_2$, this can be done easily [59–63], because the relevant mode functions are either $\sin(2\pi n\, y)$ or $\cos(2\pi n\, y)$. However, this approach becomes very cumbersome for orbifolds, like $T^6/\mathbb{Z}_3$ or $T^6/\mathbb{Z}_4$. For this reason we developed a procedure in [64] in which all calculations are preformed in the covering space of the orbifold. This method was after that formalized in [34] using orbifold projection operators, leading to orbifold trace formulae, which we reviewed in section 2.2. We can apply these results to evaluate the functional trace in the Fujikawa regularized anomaly (3.2). The result is the direct application of the trace formulae given in section 2.2. For example from (2.31) for $S^1/\mathbb{Z}_2$ we obtain:

$$\mathcal{A}[\Lambda] \;=\; \pi\int_{S^1/\mathbb{Z}_2}\Big\{\Omega^1_4(\Lambda;A,F)\,\delta(y)+(-)^p\,\Omega^1_4(\Lambda;A,F)\,\delta(y-R/2)\Big\}. \tag{3.8}$$

For the case of anti–periodic fermions, $p=1$, we see that the anomalies at both fixed points are opposite.

On the $S^1/\mathbb{Z}_2$ orbifold the anomalies either immediately cancel locally at the fixed points or after the use a bulk Chern–Simons term. Indeed, a five dimensional Chern–Simons term can be used to shift the anomaly from one fixed point to another. But the four dimensional zero mode anomaly cannot be removed by it. This is also true in seven dimensional models compactified on a warped $S^1/\mathbb{Z}_2$ orbifold [65,66]. In fact models with anomalous U(1)s at the fixed points can still be consistent provided, that there are some field in the bulk that can play a role in a five dimensional variant of the Green–Schwarz mechanism [35]. Interestingly in five dimensions this mechanism has two equivalent formulations: One can either use a vector multiplet or a tensor multiplet [67]. The variation of the Green–Schwarz action in either formulation is the same:

$$\delta_\Lambda S_{SGS} = -\int \mathrm{d}^4x \int \mathrm{d}^2\theta\ \xi_I \Lambda W^\alpha W_\alpha \delta(y) - \int \mathrm{d}^4x \int \mathrm{d}^2\theta\ \xi_I \Lambda W^\alpha W_\alpha \delta(y - R/2) + \text{h.c.}\ , \tag{3.9}$$

where W_α the Abelian version of the super gauge field strength introduced in (2.4). Its form is precisely such that Abelian and mixed U(1) non–Abelian and gravitational anomalies can be canceled. The coefficients ξ_I are proportional to the Fayet–Iliopoulos D–term tadpoles, which we encounter in the next section.

3.3 Anomaly Cancellation on Heterotic Orbifolds

For field theory models on orbifolds the matter at the fixed hyper surfaces can be placed more or less randomly, except that the theory needs to be locally anomaly free. However, modular invariant heterotic string models should lead to consistent field theory models. So if local anomaly cancellation is really a necessary constraint, heterotic orbifolds should fulfill this requirement. We investigate the local anomaly cancellation in heterotic strings on the orbifolds $T^6/\mathbb{Z}_3$ and $T^6/\mathbb{Z}_4$, using the spectra listed in tables 2.1, 2.3 and 2.2.

$T^6/\mathbb{Z}_3$ orbifold anomalies

The anomalies of heterotic models on $T^6/\mathbb{Z}_3$ are relatively simple, because this orbifold only has four dimensional fixed points. Using the trace formula for $T^6/\mathbb{Z}_3$ given in (2.33) we find the gaugino anomaly

$$\mathcal{A}_{T^6/\mathbb{Z}_3}[\Lambda] = \int_{T^6/\mathbb{Z}_3} \mathrm{d}^4x \mathrm{d}^6z\, \frac{1}{3}\Omega^1_{10}(\Lambda; A, F, R), + \frac{1}{3}\sum_s \Omega^1_{4\,\mathbf{R}_s}(\Lambda; A, F, R)\frac{1}{9}\delta(z - \mathfrak{z}_s), \tag{3.10}$$

where Ω^1_{10} defines the bulk anomaly by (3.5). At each of the 27 fixed points $\mathfrak{z}_s$ we obtain the anomaly $\Omega^1_{4|\mathbf{R}_s}$ due to four dimension matter representation $\mathbf{R}_s$. The factor 1/9 can be understood as follows: It arises from delocalized untwisted states which come with a multiplicity of 3, but only give 1/27 of a contribution at a given fixed point. In these models we only need to investigate the four dimensional anomalies, because the ten dimensional bulk anomalies are automatically canceled due to the ten dimensional Green–Schwarz mechanism. As the fixed point anomalies only involve mixed and pure U(1) anomalies, the anomaly polynomials $I_{6|s}$ have to factorize like [68]

$$\Omega_{6|s} = \alpha_s\, X_4|_s F_2|_s, \qquad X_4|_s = \mathrm{tr} R_2^2|_s - 2\sum_a \mathrm{tr} F^2_{(a)}|_s\,. \tag{3.11}$$

Here $X_{4\,GS}|_s$ denote the restrictions of $X_{4\,GS}$ to the groups G_s present at the fixed point $\mathfrak{z}_s$, the sum is over the gauge group factors in G_s, and traces $\mathrm{tr}F^2_{(a)}$ are normalized with respect to the quadratic indices of the respective gauge group factors. This means that the four dimensional version of the Green–Schwarz mechanism can be active at each of the fixed points.

Table 3.1: The $T^6/\mathbb{Z}_4$ has a six dimensional subsector on $T^4/\mathbb{Z}_2$. There are two modular invariant combinations of $\mathbb{Z}_2$ gauge shifts. The resulting gauge group and the twisted matter at a single fixed point is given.

shift V_2	gauge group $G_{[V_2]}$	untwisted matter $\mathbf{R}_{[V_2]}$	local twisted matter $\mathbf{S}_{[V_2]}$	$\mathbf{D}_{[V_2]}$
$(1;0)$	$\mathrm{E}_7\times\mathrm{SU}(2)\ \times\ \mathrm{E}_8{}'$	$(\mathbf{56},\mathbf{2})(\mathbf{1})'$	$(\mathbf{56},\mathbf{1})(\mathbf{1})'$	$4(\mathbf{1},\mathbf{2})(\mathbf{1})'$
$(1;2)$	$\mathrm{E}_7\times\mathrm{SU}(2)\ \times\ \mathrm{SO}(16)'$	$(\mathbf{56},\mathbf{2})(\mathbf{1})'+(\mathbf{1})(\mathbf{128}_s)'$	$(\mathbf{1},\mathbf{2})(\mathbf{16}_v)'$	

Table 3.2: This table gives the factorization coefficients c_i and d_i, defined by (3.13), for the two $\mathrm{E}_8\times\mathrm{E}_8{}'$ heterotic $T^6/\mathbb{Z}_4$ models $(1,0)$ and $(1,2)$, given in table 3.1.

model	G_i	E_7	$\mathrm{SU}(2)$	$\mathrm{E}_8{}'$	$\mathrm{SO}(16)'$
	c_i	1/6	2	1/30	1
$(1,0)$	d_i	1	12	-1/5	–
$(1,2)$	d_i	-1/3	28	–	2

$T^6/\mathbb{Z}_4$ orbifold anomalies

Next we investigate the more complicated case of $T^6/\mathbb{Z}_4$. For this orbifold we have to check that anomaly cancellation is possible both in four and in six dimensions. The total anomaly is given by

$$\mathcal{A}_{T^6/\mathbb{Z}_4}(\Lambda) = \int\Big\{\frac{1}{4}\cdot\frac{1}{2}\,\Omega^1_{10|\mathbf{Ad}_{[0]}} + \sum_{p,q}\frac{1}{16}\cdot\frac{1}{2}\,\big(I^1_{6|\mathbf{Ad}_{[v^2_{p\,q}]}} - \Omega^1_{6|\mathbf{R}_{[v^2_{p\,q}]}}\big)\delta^4(z-\mathfrak{z}^2_{p\,q})\mathrm{d}^4z$$
$$+\sum_{p,q}\frac{1}{16}\cdot 2\,\Omega^1_{6|\mathbf{r}_{[v^4_{p\,q}]}}\delta^6(z-\mathfrak{z}^4_{p\,q})\mathrm{d}^6z\Big\}. \tag{3.12}$$

This formula combines ten dimensional bulk anomalies, together with six and four dimensional anomalies on the various hyper surfaces, depicted in figure 2.5. As before the ten dimensional anomalies are canceled by the original ten dimensional Green–Schwarz mechanism.

For the six dimensional anomalies we find a factorized expression

$$\Omega_{8|\mathfrak{z}^2_{pq}} = -\Big[\mathrm{tr}R^2 - \sum_i c_i\mathrm{tr}(iF_i)^2\Big]\frac{-i}{(2\pi)^3}\frac{1}{16^2}\Big[\mathrm{tr}R^2 - \sum_i d_i\mathrm{tr}(iF_i)^2\Big]\ , \tag{3.13}$$

where i,j runs over the (semi–simple) gauge group factors and the traces are taken in the corresponding fundamental representations. The factors c_i give universal normalization of the quadratic traces, while the coefficients d_i are model dependent. Table 3.2 lists both c_i and d_i for both models given in table 3.1.

At the four dimensional fixed points we find, that only the models that contain U(1) factors may be anomalous. But like in ten and six dimensions the anomaly polynomial factorizes [69]:

$$\Omega_{6|\mathfrak{z}^4_{p,q}} = \Big[\mathrm{tr}R^2 - \sum_i c_i\mathrm{tr}(iF_i)^2\Big]\frac{-i}{(2\pi)^2}\frac{1}{48}\Big[F_{\mathrm{U}(1)}\,\mathrm{tr}(q_{v^4_{p\,q}}) + F_{\mathrm{U}(1)'}\,\mathrm{tr}(q'_{v^4_{p\,q}})\Big]\ , \tag{3.14}$$

Table 3.3: The resulting gauge groups and six dimensional untwisted matter representations are given for representatives of the possible $\mathbb{Z}_4$ gauge shifts on $T^6/\mathbb{Z}_4$. General gauge shifts, which have been brought to their standard form, are classified by computing $8V_4^2$. For $8V_4^2 = 4$ there are two inequivalent gauge shifts that can be distinguished by $\sum_I V_4^I \mod 2 = 0, 1$.

No. $8V_4^2$	shift $4V_4$	gauge group $\mathbf{Ad}_{[V_4]}$	untwisted matter $\mathbf{r}_{[V_4]}$	untwisted matter $\mathbf{R}_{[V_4]}$
0	(00000000)	E_8	nothing	nothing
1	(11000000)	$E_7 \times U(1)$	$(\mathbf{56})_1$	$(\mathbf{1})_2 + (\mathbf{1})_{-2}$
2	(20000000)	$SO(14) \times U(1)$	$(\mathbf{64}_s)_1$	$(\mathbf{14}_v)_2 + (\mathbf{14}_v)_{-2}$
3	(21100000)	$E_6 \times SU(2) \times U(1)$	$(\overline{\mathbf{27}}, \mathbf{2})_1 + (\mathbf{1}, \mathbf{2})_{-3}$	$(\overline{\mathbf{27}}, \mathbf{1})_{-2} + (\mathbf{27}, \mathbf{1})_2$
4_0	(22000000)	$E_7 \times SU(2)$	nothing	$(\mathbf{56}, \mathbf{2})$
4_1	(1111111-1)	$SU(8) \times U(1)$	$(\overline{\mathbf{56}})_1 + (\mathbf{8})_{-3}$	$(\mathbf{28})_2 + (\overline{\mathbf{28}})_{-2}$
5	(31000000)	$SO(12) \times SU(2) \times U(1)$	$(\mathbf{32}_s, \mathbf{1})_{-1} + (\mathbf{12}_v, \mathbf{2})_1$	$(\mathbf{32}_c, \mathbf{1})_0 + (\mathbf{1}, \mathbf{1})_2 + (\mathbf{1}, \mathbf{1})_{-2}$
6	(22200000)	$SO(10) \times SU(4)$	$(\mathbf{16}_c, \overline{\mathbf{4}})$	$(\mathbf{10}_v, \mathbf{6})$
7	(31111100)	$SU(8) \times SU(2)$	$(\overline{\mathbf{28}}, \mathbf{2})$	$(\mathbf{70}, \mathbf{1})$
8	(40000000)	$SO(16)$	nothing	$\mathbf{128}_c$

where the coefficients c_i are again related to the indices of the various groups that exist at this four dimensional fixed point. Because factorization only allows a single field strength to appear on the right hand side of (3.14), only U(1) factors may have pure and mixed anomalies. The local sum of U(1) charges decide whether a given U(1) factor is anomalous or not. In table 4.2 we have computed the sum of charges for all models with U(1) factors, listed in table 3.4. The models $(3;0)$ and $(3;4_0)$ do not have anomalous U(1) even though they contain U(1) factors. All other models considered in table 4.2 have only one anomalous U(1), except for model $(2;5)$. However, as observed in ref. [68], one can always find two other linear combinations of the charges $q_{v^4_{p\,q}}$ and $q'_{v^4_{p\,q}}$, such that only one of them is anomalous.

3.4 Conclusions

The study of anomalies is crucial to determine whether a theory can make sense at the quantum level or not. Anomalies are generated both at the orbifold singularities and in the bulk of the orbifold. To prove this we used Fujikawa's method to compute anomalies together with the orbifold trace formula developed in the previous section. The analyses of various heterotic orbifolds confirmed the expectation that irreducible non–Abelian anomalies cancel locally, and therefore, in particular, at the orbifold fixed points. Anomalies in U(1) gauge symmetries are canceled at the various orbifold fixed points using four and six dimensional Green–Schwarz mechanisms.

Table 3.4: There are 12 modular invariant combinations of $\mathbb{Z}_4$ $E_8 \times E_8{}'$ gauge shifts, which are listed in table 3.3. (The numbers $(n; n')$ correspond to the first column of that table.) The resulting gauge group and the single and double twisted matter at a single fixed point is given.

shift V_4	gauge group $G_{[V_4]}$	single twisted $\mathbf{T_1}_{[V_4]}$	double twisted $\mathbf{T_2}_{[V_4]}$
$(3;0)$	$E_6 \times SU(2) \times U(1) \times$ $E_8{}'$	$(\overline{\mathbf{27}},\mathbf{1})_{-1/2}(\mathbf{1})' + 2(\mathbf{1},\mathbf{2})_{-3/2}(\mathbf{1})'$ $+5(\mathbf{1},\mathbf{1})_{3/2}(\mathbf{1})'$	$(\overline{\mathbf{27}},\mathbf{1})_{1}(\mathbf{1})' + (\mathbf{1},\mathbf{1})_{-3}(\mathbf{1})'$ $+2(\mathbf{1},\mathbf{2})_{0}(\mathbf{1})'$
$(3;4_0)$	$E_6 \times SU(2) \times U(1) \times$ $E_7{}' \times SU(2)'$	$(\mathbf{1},\mathbf{2})_{-3/2}(\mathbf{1},\mathbf{2})' + 2(\mathbf{1},\mathbf{1})_{3/2}(\mathbf{1},\mathbf{2})'$	$(\mathbf{27},\mathbf{1})_{-1}(\mathbf{1},\mathbf{1})' + (\mathbf{1},\mathbf{1})_{3}(\mathbf{1},\mathbf{1})'$ $+2(\mathbf{1},\mathbf{2})_{0}(\mathbf{1},\mathbf{1})'$
$(3;4_1)$	$E_6 \times SU(2) \times U(1) \times$ $SU(8)' \times U(1)'$	$(\mathbf{1},\mathbf{1})_{3/2}(\mathbf{8})'_{-1} + (\mathbf{1},\mathbf{2})_{-3/2}(\mathbf{1})'_{2}$ $+2(\mathbf{1},\mathbf{1})_{3/2}(\mathbf{1})'_{2}$	$(\mathbf{1},\mathbf{2})_{0}(\overline{\mathbf{8}})'_{-1}$
$(3;8)$	$E_6 \times SU(2) \times U(1) \times$ $SO(16)'$	$(\mathbf{1},\mathbf{1})_{3/2}(\mathbf{16}_v)'$	$(\overline{\mathbf{27}},\mathbf{1})_{1}(\mathbf{1})' + (\mathbf{1},\mathbf{1})_{-3}(\mathbf{1})'$ $+2(\mathbf{1},\mathbf{2})_{0}(\mathbf{1})'$
$(7;0)$	$SU(8) \times SU(2) \times$ $\times E_8{}'$	$(\overline{\mathbf{8}},\mathbf{2})(\mathbf{1})' + 2(\mathbf{8},\mathbf{1})(\mathbf{1})'$	$(\mathbf{28},\mathbf{1})(\mathbf{1})' + 2(\mathbf{1},\mathbf{2})(\mathbf{1})'$
$(7;4_0)$	$SU(8) \times SU(2) \times$ $E_7{}' \times SU(2)'$	$(\mathbf{8},\mathbf{1})(\mathbf{1},\mathbf{2})'$	$(\overline{\mathbf{28}},\mathbf{1})(\mathbf{1},\mathbf{1})' + 2(\mathbf{1},\mathbf{2})(\mathbf{1},\mathbf{1})'$
$(7;4_1)$	$SU(8) \times SU(2) \times$ $SU(8)' \times U(1)'$	$(\mathbf{8},\mathbf{1})(\mathbf{1})'_{2}$	$(\mathbf{1},\mathbf{2})(\mathbf{8})'_{1}$
$(7;8)$	$SU(8) \times SU(2) \times$ $SO(16)'$	nothing	$(\mathbf{28},\mathbf{1})(\mathbf{1})' + 2(\mathbf{1},\mathbf{2})(\mathbf{1})'$
$(2;1)$	$SO(14) \times U(1) \times$ $E_7{}' \times U(1)'$	$(\mathbf{14}_v)_{-1}(\mathbf{1})'_{1/2} + (\mathbf{1})_{1}(\mathbf{1})'_{-3/2}$ $+5(\mathbf{1})_{1}(\mathbf{1})'_{1/2}$	$(\mathbf{14}_v)_{0}(\mathbf{1})'_{1} + (\mathbf{1})_{2}(\mathbf{1})'_{-1}$ $+(\mathbf{1})_{-2}(\mathbf{1})'_{-1}$
$(2;5)$	$SO(14) \times U(1) \times$ $SO(12)' \times SU(2)' \times U(1)'$	$(\mathbf{1})_{1}(\mathbf{12}_v,\mathbf{1})_{1/2} + 2(\mathbf{1})_{1}(\mathbf{1},\mathbf{2})_{-1/2}$	$(\mathbf{14}_v)_{0}(\mathbf{1},\mathbf{1})_{1} + (\mathbf{1})_{2}(\mathbf{1},\mathbf{1})_{-1}$ $+(\mathbf{1})_{-2}(\mathbf{1},\mathbf{1})_{-1}$
$(6;1)$	$SO(10) \times SU(4) \times$ $E_7{}' \times U(1)'$	$(\mathbf{16}_c,\mathbf{1})(\mathbf{1})'_{1/2} + 2(\mathbf{1},\mathbf{4})(\mathbf{1})'_{1/2}$	$(\mathbf{10}_v,\mathbf{1})(\mathbf{1})'_{-1} + (\mathbf{1},\mathbf{6})(\mathbf{1})'_{1}$
$(6;5)$	$SO(10) \times SU(4) \times$ $SO(12)' \times SU(2)' \times U(1)'$	$(\mathbf{1},\mathbf{4})(\mathbf{1},\mathbf{2})'_{-1/2}$	$(\mathbf{10}_v,\mathbf{1})(\mathbf{1},\mathbf{1})'_{-1} + (\mathbf{1},\mathbf{6})(\mathbf{1},\mathbf{1})'_{1}$

Chapter 4

Stability: Tadpoles

In the previous section we have investigate which orbifold theories lead to consistent models. In this section we assume that our theories are free of dangerous non–Abelian anomalies, and continue the investigation of fields and strings on orbifolds. Now we would like to see whether such theories are stable. Instabilities for us means that some fields develop large VEVs by quantum corrections. The root cause for such instabilities are quadratically divergent D–terms.

One of the main motivations for introducing supersymmetry is that all quadratic divergences are absent. However, there is one exception: The auxiliary field D of the U(1) vector multiplet can be quadratically divergent. This happens when the sum of charges of all chiral superfields does not vanish [70], i.e. only when the theory possesses a mixed gravitational–U(1) anomaly. As discussed in the previous section this type of anomaly can be canceled by a Green–Schwarz mechanism. Nevertheless the mass of a scalar becomes quadratically divergent after the auxiliary D field has been removed.

In the next subsection we see that these instabilities appear on orbifolds as delta–like singularities. As string theory is often considered to be a finite non–singular theory, we investigate how these field theory singularities are smoothed out by stringy effects.

Figure 4.1: The indicated contour allows one to replace a sum over Kaluza–Klein masses by a complex integral. The symbols "X" denote the positions of poles f, and the dots • indicate the poles of $\mathcal{P}$.

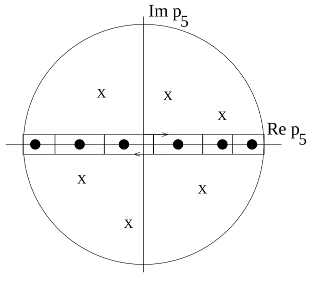
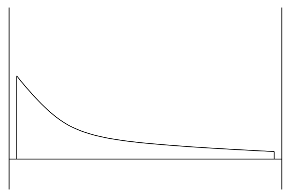


Table 4.1: The two basic shapes (eqs. (4.2) and (4.3)) of the zero mode with charge q_b are displayed for a finite value of the cut-off Λ. Delta function localizations, denoted by the arrows, happens if the signs of the FI–parameter ξ_0'' and charge q_b are the same. When the signs are opposite, the wave function falls off exponentially and vanishes at both branes.

$\xi_0'' q_b > 0$	$\xi_0'' q_b < 0$
 0 πR	 0 πR

4.1 Gauge Field Tadpoles on Orbifolds

The importance of Fayet–Iliopolous tadpoles for orbifold model building was first realized in [71]. This paper was a reaction to the claim, that it possible to construct an orbifold model which is a five dimensional version of the standard model but with one parameter less [30, 72]. We showed that it contains at least one additional counter term more than was assumed: the localized Fayet–Iliopolous terms at the orbifold fixed points. The method employed to compute the existence of a Fayet–Iliopolous tadpole was to replace the sum over a Kaluza–Klein tower of states, by a contour integral [73]:

$$\sum_{n\in\mathbb{N}} f(m_n) = \frac{1}{2\pi i}\int_{\ominus} \mathrm{d}p_5\,\mathcal{P}(p_5) f(p_4, p_5).$$

The figure 4.1 gives a schematic picture of this integral in the complex p_5-plane. The function $\mathcal{P}$ is chosen such that it has unit residue at it single poles located at the Kaluza–Klein masses. This procedure was used to compute the effective potential within universal extra dimensions [74], and has been extend to five dimensional warped geometries in [75].

In a series of papers [60,61,76] we subsequently showed that the localized tadpoles for the D terms are accompanied by tadpoles for the physical scalar ϕ in the five dimensional vector multiplet:

$$< \partial_y \phi > \;=\; g_5 \sum_{I=0,1} \Big(\xi_I \,+\, \xi_I''\,\partial_y^2\Big)\Big[\delta(y) + \delta(y - I\,R/2)\Big] \tag{4.1}$$

where $\xi_I = \frac{\Lambda^2}{16\pi^2}\big(\frac{\mathrm{tr}\,q}{2} + \mathrm{tr}\,q_I\big)$ and $\xi_I'' = \frac{1}{4}\frac{\ln\Lambda^2}{16\pi^2}\frac{\mathrm{tr}\,q}{2}$. The tadpole gives rise to a non–trivial profile for this physical scalar, which in turn leads to strong localization effects of bulk states towards the orbifold fixed points: When the charge q_b of a bulk field and the FI–parameter ξ_0'' have the same sign, we obtain the zero mode profile

$$\phi_{0+}^2(y) = \frac{2e^{g_5 q_b \xi_0\, y}}{1 + e^{g_5 q_b \xi_0\,\pi R}}\left[\delta(y) + \delta(y - \pi R)\right], \qquad\qquad \xi_0'' q_b > 0. \tag{4.2}$$

Table 4.2: The sum of charges of heterotic $E_8 \times E_8$ $T^6/\mathbb{Z}_4$ models with U(1) factor(s) is computed for each matter sector ($\mathbf{r}_{[V_4]}$, $\mathbf{T2}_{[V_4]}$ and $\mathbf{T1}_{[V_4]}$) separately. (If a model contains two U(1), the bracket indicate the sum of charges of the U(1)'s in the E_8 and $E_8{}'$ sectors.) The sum of these three contributions determines whether that U(1) is anomalous or not.

	models with U(1) factors								
	$(3;0)$	$(3;4_0)$	$(3;4_1)$	$(3;8)$	$(7;4_1)$	$(2;1)$	$(2;5)$	$(6;1)$	$(6;5)$
$\frac{2}{16}\mathrm{tr}_{\mathbf{r}_{[V_4]}}(q)$	6	6	(6, 4)	6	4	(8, 7)	(8, -1)	7	-1
$\frac{1}{4}\mathrm{tr}_{\mathbf{T2}_{[V_4]}}(q)$	6	-6	(0, -4)	6	4	(0, 3)	(0, 3)	-1	-1
$\mathrm{tr}_{\mathbf{T1}_{[V_4]}}(q)$	-12	0	(12, 0)	24	16	(-8, 8)	(16, 4)	12	-4
sum	0	0	(18, 0)	36	24	(0, 18)	(24, 6)	18	-6

Hence the zero mode has the delta function support on the two fixed points, but the height at these two fixed points is not the same. In the second case, we find that the zero mode does not live on the boundaries of the interval

$$\phi_{0+}^2(y) = \frac{g_5 q_b \xi_0\, e^{g_5 q_b \xi_0\, y}}{e^{g_5 q_b \xi_0\, \pi R} - 1} \begin{cases} 1 & 0 < y < \pi R, \\ 0 & y = 0, \pi R, \end{cases} \qquad \xi_0'' q_b < 0. \tag{4.3}$$

Figure 4.1 schematically pictures these (non–)localization effects. Such physical effects were also studied in [77, 78], and also appear in six dimensions [79]. Extension to warped spaces have been investigated in [80, 81].

In six and higher dimensions these tadpoles become tadpoles for the internal part of U(1) gauge field strengths [82]. On the orbifold $T^6/\mathbb{Z}_3$ we find the expression

$$L_{FI} = -\xi_I\, F^I_{\underline{A}A}, \qquad \xi_I = \sum_s \left(\frac{\Lambda^2}{16\pi^2}\, \mathrm{tr}_{\mathbf{L}_s}(H_I) + \frac{1}{27}\frac{\ln \Lambda^2}{16\pi^2}\, \mathrm{tr}_{\mathbf{R}_s}(H_I)\Delta \right) \delta(z - \mathfrak{z}_s - \Gamma). \tag{4.4}$$

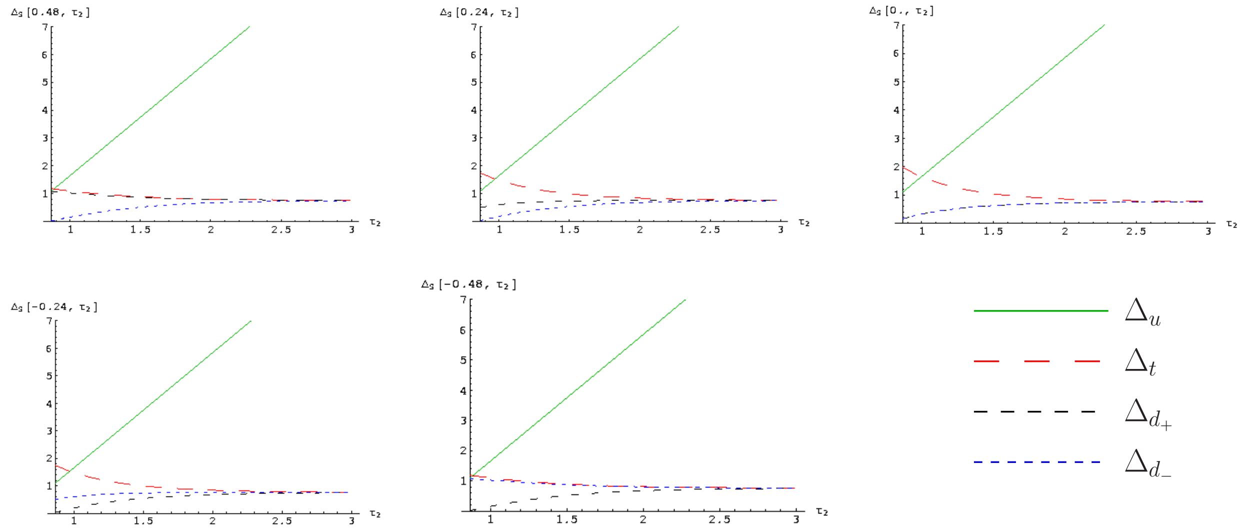
On the orbifold $T^6/\mathbb{Z}_4$ the expression for the tadpole becomes

$$L_{FI} = F^I_{\underline{A}A} \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \left\{ \frac{\frac{2}{16}\mathrm{tr}_{\mathbf{r}_{[v^4_{st}]}}(H_I)}{p^2 + \frac{1}{4}\Delta_1 + \frac{1}{2}\Delta_{23}} + \frac{\frac{1}{4}\mathrm{tr}_{\mathbf{T2}_{[v^4_{st}]}}(H_I)}{p^2 + \frac{1}{4}\Delta_1} + \frac{\mathrm{tr}_{\mathbf{T1}_{[v^4_{st}]}}(H_I)}{p^2} \right\} \delta^6(z - \mathfrak{z}^4_{st}). \tag{4.5}$$

The factors 1/4 and 1/2 in front of the internal Laplacian $\Delta_1 = \bar{\partial}_1 \partial_1$ and $\Delta_{23} = \bar{\partial}_2 \partial_2 + \bar{\partial}_3 \partial_3$ are consequences of the trace formula (2.35) and similar results on fixed $T^2/\mathbb{Z}_2$. We have written this expression as traces of H_I, such that the relative contributions of the different terms at each of the fixed points of $T^6/\mathbb{Z}_4$ can directly be read off from table 4.2.

All these results rely on the implicit assumption that field theory can be trusted near the orbifold fixed point even though the tadpoles have delta–like support and are highly divergent. This very strongly peaked behavior might be physically somewhat suspicious, because in nature we do not expect to encounter truly singular properties. To shed more light on their true status we preform a string analysis of these tadpoles in the next subsection.

Figure 4.2: The five plots show the differences of the correlators $\Delta_s(\tau_1, \tau_2)$ for the values $\tau_1 = -0.48, -0.24, 0, 0.24$ and 0.48 in the different sectors $s = u, t, d_+$ and d_-.



4.2 String Resolution of Tadpole Singularities

As we showed in the previous subsection the tadpoles for the internal parts of the gauge field strengths computed in field theory are singular functions at the orbifold fixed points. For anomalies we found similar delta–like localization. But since anomalies are directly related to topological data, such behavior neither surprising nor problematic. For physical observables one expect that such singular behavior would be smoothed out in a UV complete theory. As string theory is supposed to be UV complete, it is interesting to see how these localized tadpoles appear in string theory. We perform this investigation in $\mathrm{E}_8 \times \mathrm{E}_8$ heterotic string on $\mathbb{C}^3/\mathbb{Z}_3$. This computation extends the D–term tadpoles in heterotic strings considered in [83–85]. (For similar computations on the type–I side see [86].)

To be able to make an easy comparison, we first represent the field theoretical result of the tadpole as

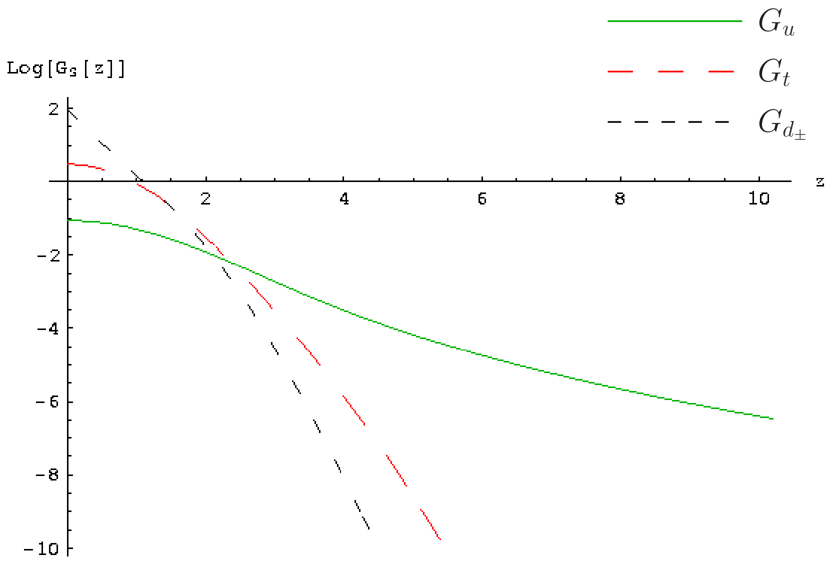
$$\langle F^b_{j\underline{j}}(k)\rangle = \frac{\delta^4(k_4)}{(2\pi)^4}\,\frac{\pi}{4}\Lambda^2 \int\limits_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{d}\tau_1 \int\limits_{1}^{\infty} \frac{\mathrm{d}\tau_2}{\tau_2^2} \sum_{s=un,tw} Q^b_s\, e^{-\Delta_s\, k_i k_{\underline{i}}/\Lambda^2}. \tag{4.6}$$

Here the quantities $Q^b_{un}, Q^b_{tw}, \Delta_{un}$ and Δ_{tw} are given by

$$Q^b_{un} = \frac{3}{27}\,\mathrm{tr}_{\mathbf{R}}(q_b), \qquad Q^b_{tw} = \mathrm{tr}_{\mathbf{S}}(q_b) + 3\,\mathrm{tr}_{\mathbf{T}}(q_b), \qquad \Delta_{un} = 4\pi\,\tau_2\,\frac{1}{3}, \quad \Delta_{tw} = 0, \tag{4.7}$$

in field theory. The subscripts un and tw refer to the bulk and localized contributions, which in a stringy language are called untwisted and twisted sectors, respectively. For the untwisted sector, $\Delta_{un} \neq 0$, and the tadpole takes the form of Gaussian distributions with widths depending on τ_2. This variable is integrated up to the cut–off of the effective field theory, which has been scaled to unity by

Figure 4.3: The spatial extension of the tadpole contributions in the sectors u, t and $d_\pm$ are displayed on a logarithmic scale. The curve for the u–sector fall off much slower than the others for large values of $|z|$, as only it corresponds to bulk contributions.



pulling out Λ. For large internal momenta k_6 the untwisted tadpole is strongly damped. For the fixed point states there is no such suppression because $\Delta_{tw} = 0$. This reflects that in field theory we treat the twisted states as localized finitely strongly at the fixed points.

We now turn to the heterotic sting results. The gauge field tadpole becomes a sum over the different sectors $s = u, t, d_\pm$

$$\langle F^b_{j\underline{j}}(k_6)\rangle = \frac{\delta^4(k_4)}{(2\pi)^4} \sum_{s=u,t,d_\pm} G^b_s(k_6), \quad G^b_s(k_6) = \int_{\mathcal{F}} \frac{\mathrm{d}^2\tau}{\tau_2^2}\, Q^b_s(\tau)\, e^{-\Delta_s(\tau,\bar\tau)\, k_i k_{\underline{i}}}. \tag{4.8}$$

In this expression $Q^b_s(\tau)$ can be thought of as the trace of the q_b charges of sector s. In string theory the orbifold space coordinates X^i and $X^{\underline{i}}$ are fields on the torus world sheet. Their correlator $\Delta_s(\tau,\bar\tau) = \langle X^{\underline{i}} X^i\rangle_s(\tau,\bar\tau)$ sets the width of the orbifold singularity in sector s, because it appears as the (inverse) standard deviation of the Gaussian distribution for the internal momentum k_6. The exact one loop string expression for the functions $Q^b_s(\tau)$ and $\Delta_s(\tau,\bar\tau)$ have been obtained in ref. [87–89]. We plotted the functions $\Delta_s(\tau_1,\tau_2)$ for the different sectors $s = u, t, d_+$ and d_- in figure 4.2 for five values τ_1. The behavior of the untwisted states (u) is very different from that of the twisted states (t, and $d_\pm$) for large τ_2: The former grow linearly with τ_2, while the latter all approach the constant c_0.

In figure 4.3 we plotted the profiles as a function of the radial variable $|z|$ in the six internal dimensions. Close to the orbifold singularity the twisted states in the t and $d_\pm$–sectors dominate the tadpole. At a distance of about $z = 2.5$ string lengths all three sectors contribute with comparable magnitude. The profiles shows pronounced differences between the drop off in the sectors: The curves corresponding to the t and $d_\pm$–sectors fall of much faster than the one for the u–sector. These differences are consistent with our understanding that the u–sector corresponds to bulk states, while the t and $d_{\pm}$–sectors constitute the fixed point states.

4.3 Conclusions

The Fayet–Iliopolous D–term has been the achille's heal of supersymmetry being the only quantity which is quadratically divergent. In orbifold theories in extra dimensions their generalizations again spell trouble, and lead to delta–like quadratically divergent singularities at the orbifold fixed points. In five (and higher) dimensions they can lead to strong localization effect of bulk states towards the fixed points. These effects are real though smoothed out by stringy corrections.

Chapter 5

Perturbativity: Gauge Couplings

The strength of gauge interactions is measured by the gauge couplings. In four dimensional theories the properties of gauge couplings are well–known. One of their main properties is that they depend on the renormalization scale. This dependence is encoded in renormalization group equations, which can be computed perturbatively. In four dimensional supersymmetry theories only self energy diagrams give corrections to the gauge coupling running.

The story becomes more complicated in extra dimensions and in particular on orbifolds. The reason is that such gauge theories are characterized by more than one coupling: There is the bulk gauge coupling and gauge couplings at the orbifold fixed points. In addition higher derivative operators arise because the higher dimensional theory is non–renormalizable. It is these effects that we would like to investigate. Therefore we compute the operators that are needed as one loop counter terms rather than just the couplings. In this way we do not unintentionally ignore additional effects.

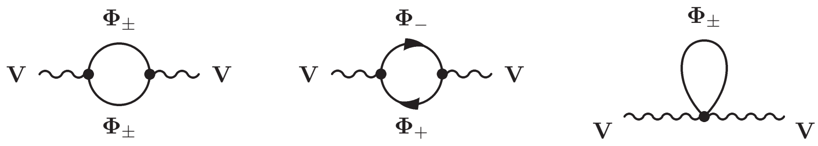
5.1 Local Gauge Coupling Running on Orbifolds

We compute the renormalization of gauge couplings on orbifolds using the orbifold compatible fields developed in section 2.2. As reviewed in section 2.1 a hyper multiplet in five or six dimensions can be described by two oppositely charged chiral multiplets Φ_+ and Φ_-. The vector multiplet consists of a four dimensional vector V and a chiral adjoint S superfields. Hence, we can use the supergraph techniques reviewed in 2.1 to compute of gauge coupling running in higher dimensional supersymmetric theories on orbifolds. As the details were reported in [19, 20] we only give some relevant diagrams and results.

As the quantum corrections on $T^2/\mathbb{Z}_N$ are to a large extend the same as the $S^1/\mathbb{Z}_2$ (because the diagrams, see figure 5.1, for a hyper multiplet are identical in five and six dimensions), we refrain from giving the five dimensional results here. However, there are two additional effects in six dimensions: First of all the orbifold action can be larger than just $\mathbb{Z}_2$. For even ordered orbifolds there exist $\mathbb{Z}_2$ and non–$\mathbb{Z}_2$ fixed points. The $\mathbb{Z}_2$ fixed points have similar properties as the fixed points of $S^1/\mathbb{Z}_2$, while the contributions at non–$\mathbb{Z}_2$ fixed points are fundamentally different. At the $\mathbb{Z}_2$ fixed points we obtain

$$\mathcal{S}^{\text{hyper}}_{\mathbb{Z}_2} = \frac{-1}{(4\pi)^2 N}\Big(\frac{1}{\bar\epsilon} + \ln\frac{\mu^2}{m^2}\Big)\int d^6x\, d^4\theta\, \mathrm{tr} Z_+^{N/2}\Big(\big[\overline{S} - \sqrt{2}\bar\partial V, S - \sqrt{2}\partial V\big] - \big[\bar\partial V, \partial V\big]\Big)\delta^2\big(2z\big), \quad (5.1)$$

Figure 5.1: The gauge multiplet receives VV self energy corrections from the hyper multiplet. The proper self energy graphs are the first two diagrams. The tadpole graph is the last diagram.



here the matrix $Z_+ = e^{iA_+}$ defines the hyper multiplet boundary condition: $\Phi_+(\theta z) = Z_+\,\Phi(z)$. At the non–$\mathbb{Z}_2$ fixed points the result reads

$$\begin{aligned}\mathcal{S}^{\text{hyper}}_{\text{non-}\mathbb{Z}_2} = \frac{-2}{(4\pi)^2 N}\Big(\frac{1}{\epsilon}+\ln\frac{\mu^2}{m^2}\Big)\sum_{b=1}^{[N/2]_*}\int d^6x\; d^4\theta\;\text{tr}\Big[&-\big(\cos b(A_+ +\varphi)+\cos bA_+\big)V\Box P_o V+\\ &+\cos\; b(A_+ +\varphi)\big(\partial V\bar\partial V-\sqrt{2}\;\partial V\overline{S}-\sqrt{2}\;S\bar\partial V+S\overline{S}\big)\\ &+\cos\;(bA_+)\big(\bar\partial V\partial V-\sqrt{2}\;\overline{S}\partial V-\sqrt{2}\;\bar\partial VS+\overline{S}S\big)\Big]\delta^2\big((1-e^{ib\varphi})z\big).\end{aligned} \tag{5.2}$$

Similar results can be obtained for contributions due to vector multiplets in the loop, see [20]. In the next subsection we perform this analysis for the vector multiplet ten dimensions.

The second effect has to do with the fact that the six dimensional theory in non–renormalizable and hence higher dimensional operators can be generated:

$$\begin{aligned}\mathcal{S}^{\text{hyper}}_{\text{bulk}} =&\frac{-2m^2}{(4\pi)^3 N}\Big(\frac{1}{\epsilon}+\ln\frac{\mu^2}{m^2}+1\Big)\int d^6x d^4\theta\;\text{tr}\Big[-V\;\Box P_0\;V+\partial V\bar\partial V-\sqrt{2}(\partial V\overline{S}+\bar\partial VS\;)+S\overline{S}\Big]+\\ &+\frac{1}{3\;(4\pi)^3 N}\Big(\frac{1}{\epsilon}+\ln\frac{\mu^2}{m^2}\Big)\int d^6x d^4\theta\;\text{tr}\Big[-V\;\Box P_0(\Box+\partial\bar\partial)\;V+\partial V(\Box+\partial\bar\partial)\bar\partial V+\\ &-\sqrt{2}\;\partial V(\Box+\partial\bar\partial)\overline{S}-\sqrt{2}\;\bar\partial V(\Box+\partial\bar\partial)S+S(\Box+\partial\bar\partial)\overline{S}\Big].\end{aligned} \tag{5.3}$$

Possible consequences of such higher derivative operators have been studied in the literature [90–95].

5.2 Local Running Couplings on Heterotic Orbifolds

The computation of local gauge couplings presented above for six dimensional orbifolds can be extended to heterotic orbifold models. The above discussion as preformed in the field theory context, but there may be additional stringy effects. We have investigated this issue for the effective four dimensional gauge couplings in ref. [96] and found, that the purely field theoretical computation is able to reproduce the full string result [97–99] upto contributions due to string winding modes, which clearly have no interpretation in field theory. The other difference is that the full string result is finite, while the field theory computation requires regularization. However, the comparison of the theoretical computation of gauge couplings and experimental measurements is perform using the field theory language. This amounts to divide the full finite string result into contributions of the four dimensional zero modes and the other states. The scale at which this splitting is performed is to a large extend arbitrary, which reintroduces a renormalization scale dependence in the string theory predictions. In light of this we assume that the most important gauge corrections in heterotic orbifold models can be captured using ten dimensional super Yang–Mills theory alone.

In this subsection we compute the local gauge couplings on the heterotic $T^6/\mathbb{Z}_4$ orbifold. For geometrical details of this orbifold we refer to section 2.2 and ref. [26]. In the $T^6/\mathbb{Z}_4$ orbifold we have to distinguish six dimensional contributions which are localized at the fixed torus (the contributions on the second line of (2.35)), and contributions which are localized at the four dimensional fixed points (the second term on the first line of (2.35)). The six dimensional fixed torus supports a six dimensional amplitude whose divergence is the same as that of the bulk contribution in the six dimensional calcuation discussed in previous subsection. The amplitude at the fixed points gives rise to the usual four dimensional divergences.

Bulk renormalization

The bulk result is obtained from the first term of (2.35) and vanishes identically. This result was to be expected, since it is known that there is no gauge coupling renormalization in ten dimensional super Yang–Mills theories. The presence of the new graph, the first graph on the second line of figure 5.2 (that is constructed from the new propagators specific to ten dimensions), is indespensable for this result to hold. Moreover, also no higher dimensional operators are generated in the bulk.

Fixed torus renormalization in six dimensions

The counter term localized at the six dimensional fixed torus in the first complex plane is a sum of a gauge kinetic term of a six dimensional gauge multiplet and a higher derivative operator. The first reads

$$\Delta\mathcal{S}^{\text{gauge}} = \frac{m^2\mu^{-2\epsilon}}{(4\pi)^3\epsilon}\int d^{10}x\,\Big\{\int d^2\theta\,\mathrm{tr}_{\mathbf{Ad}}\Big[\frac{1}{4}\,Q^{-2}W^\alpha W_\alpha\Big] + \text{h.c.} +$$
$$+\int d^4\theta\,\mathrm{tr}_{\mathbf{Ad}}\Big[Q^{-2}\Big(\Big(-\tfrac{1}{\sqrt{2}}\,\partial_1+S_1\Big)\,e^{2V}\Big(\tfrac{1}{\sqrt{2}}\,\bar\partial_1+\overline{S}_1\Big)\,e^{-2V}+\frac{1}{4}\,\partial_1 e^{-2V}\,\bar\partial_1 e^{2V}\Big)\Big]\Big\}$$
$$\times\delta^2\big((1-e^{i2\varphi_2})z_2\big)\,\delta^2\big((1-e^{i2\varphi_3})z_3\big), \tag{5.4}$$

where Q defines the orbifold conditions for the vector multiplet $V^I(\theta z) = Q^I{}_J\,V^J(z)$. We have displayed the full expression proportional to the action of a six dimensional gauge multiplet, even though, we have only calculated explicitly those terms that involve only the four dimensional gauge multiplet V and its derivatives. In order for the theory to reproduce the complete action for the six dimensional gauge multiplet, also the other terms have to be present. In addition, a higher derivative operator is localized at the fixed torus

$$\Delta\mathcal{S}^{\text{gauge}}_{\text{HDO}} = -\frac{\mu^{-2\epsilon}}{6(4\pi)^3\epsilon}\int d^{10}x\,\Bigg\{\int d^2\theta\,\text{tr}_{\mathbf{Ad}}\Bigg[\frac{1}{4}\,Q^{-2}W^\alpha\big(\Box+\partial_1\bar\partial_1\big)W_\alpha\Bigg]+\text{h.c.}+$$
$$+\int d^4\theta\,\text{tr}_{\mathbf{Ad}}\Bigg[Q^{-2}\Bigg(\Big(-\tfrac{1}{\sqrt{2}}\,\partial_1+S_1\Big)\,e^{2V}\Big(\Box+\partial_1\bar\partial_1\Big)\Big(\tfrac{1}{\sqrt{2}}\,\bar\partial_1+\overline{S}_1\Big)\,e^{-2V}+$$
$$+\frac{1}{4}\,\partial_1\,e^{-2V}\big(\Box+\partial_1\bar\partial_1\big)\,\bar\partial_1\,e^{2V}\Bigg)\Bigg]\Bigg\}\;\delta^2\big((1-e^{i2\varphi_2})z_2\big)\,\delta^2\big((1-e^{i2\varphi_3})z_3\big). \tag{5.5}$$

Fixed points renormalization in four dimensions

Since the delta functions $\prod_{J=1}^3\delta^2\big((1-e^{i\varphi_J})z_J\big)$ and $\prod_{J=1}^3\delta^2\big((1-e^{-i\varphi_J})z_J\big)$ describe the same fixed point set, we find at the four dimensional fixed points

$$\Delta\mathcal{S}^{\text{gauge}}_{\text{fp}} = \frac{\mu^{-2\epsilon}}{(4\pi)^2\epsilon}\int d^{10}x\,\Bigg\{\int d^2\theta\,\text{tr}_{\mathbf{Ad}}\Bigg[\frac{1}{4}\,Q^{-1}W^\alpha W_\alpha\Bigg]+\text{h.c.}\Bigg\}\prod_{J=1}^{3}\delta^2\big((1-e^{i\varphi_J})z_J\big). \tag{5.6}$$

Obtaining complete gauge coupling corrections

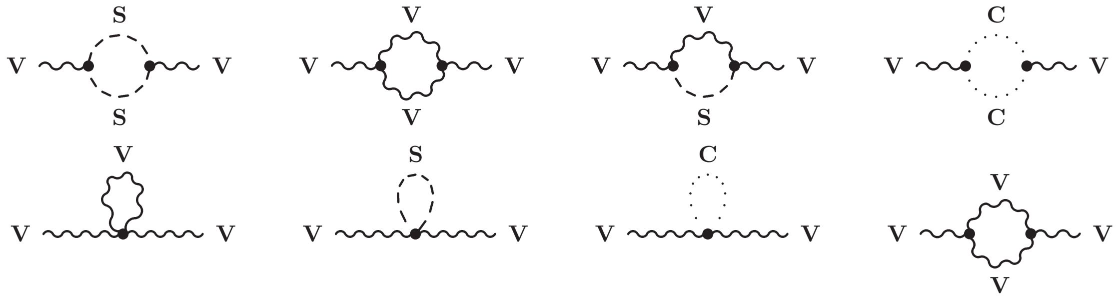
In the discussion above we have determined the contributions to the VV selfenergy from the ten dimensional super Yang–Mills theory and from six dimensional hyper multiplets. To compute the complete contribution to the gauge couplings one has to combine these results: The ten dimensional bulk is completely treated. At the six dimensional fixed points we have to add the localized contribution of the ten dimensional gauge theory with the effects of the twisted hyper multiplets localized there. The four dimensional fixed points get contributions from three sources: the ten dimensional gauge fields, the six dimensional hyper multiplets, and the four dimensional chiral multiplets. When doing so, one has to take care of the relative multiplicity factors that arise from the orbifold trace formulae. The required analysis is therefore similar to the way one obtains the local anomalies in the bulk, fixed surfaces and points of orbifolds, see section 3.3.

The computations reported here only focused on the VV self energy of the ten dimensional vector multiplet. But as mentioned before, this multiplet also contains three chiral superfields S_1, S_2 and S_3. They have self energies and mixed self energies that also all renormalize. This means that one has to compute the corresponding diagrams for those superfields as well. It should be realized that at the six and four dimensional fixed points these chiral multiplets partly have a different interpretation: They do not all constitute gauge degrees of freedom there, but should rather be viewed as untwisted matter. At the six dimensional fixed points, S_1 and V form a six dimensional vector multiplet, while S_2 and S_3 untwisted hyper multiplets. At the four dimensional fixed points all three of the chiral adjoints are reinterpreted as untwisted chiral matter. The computation of their wavefunction renormalization has only been partly performed on orbifolds locally in the literature.

5.3 Conclusions

The gauge coupling is a central quantity in a supersymmetric gauge theory that determines its properties. On orbifolds gauge theories are necessarily described by a multitude of gauge couplings: Aside

Figure 5.2: Most vector multiplet self energy graphs in ten dimensions are straightforward extensions of those in lower dimensions. Only the first graph on the second line appears in the ten dimensions.



from the ordinary bulk gauge coupling, a new coupling arises that parameterizes the relevance of higher derivative corrections to the bulk action. In addition, gauge coupling renormalizations occur at the orbifold fixed points. We demonstrate that by using orbifold compatible (supersymmetric) field theories all these corrections can be systematically computed even for heterotic orbifold models.

Chapter 6

Deformations: Blowup of Singularities

As noted in the introduction field theories on orbifolds cannot be trusted because of curvature and other singularities at the orbifold fixed points. Even though we have shown that for some computations we are still able to make sense of the results, a more physical approach would be to smooth out the orbifold fixed points, so that quantities like the curvature do not diverge in the first place. Preserving a certain amount of supersymmetry in this process implies that the resulting smooth space needs to be rather complicated, i.e. is a compact Calabi–Yau space. (Analysis of the spectra on smooth compact Calabi–Yaus have been performed in [100–104].) To simplify the analysis we focus on a single fixed point and consider its non–compact Calabi–Yau blowup. In the first subsection we show how to construct explicit $\mathbb{C}^n/\mathbb{Z}_n$ blowups, and explain how one can obtain four dimensional models from them. Using these blowup models essentially the complete untwisted and twisted spectra of heterotic orbifold models can reproduced. For more complicated orbifold singularities unfortunately no explicit blowups are known, however, one can describe topological properties using toric geometry as explained in the final subsection.

6.1 Explicit $\mathbb{C}^n/\mathbb{Z}_n$ Blowups

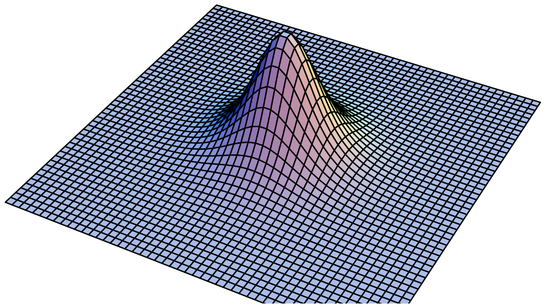
We review the explicitly construction of a blowup of the $\mathbb{C}^n/\mathbb{Z}_n$ orbifold with possible U(1) bundles following [105, 106]. The blowup is defined as the cone that is the nth power of the fundamental complex line over $\mathbb{CP}^{n-1}$. For a detailed discussion of the Kähler geometry of $\mathbb{CP}^{n-1}$ and its complex line bundles, see [107–109]. By requiring Ricci–flatness we obtain the resolution manifold $\mathcal{M}^n$, that we are looking for [110]. Similar constructions for more general coset spaces than $\mathbb{CP}^{n-1}$ can be found in [111–113].

The $\mathbb{C}^n/\mathbb{Z}_n$ orbifold is defined by the $\mathbb{Z}_n$ action $\widetilde{Z} \rightarrow \theta\, \widetilde{Z}$, where $\theta = e^{2\pi i\, \phi}$, with $\phi = (1, \ldots, 1)/n$. The geometry of the non–singular blowup is described by the Kähler potential

$$\mathcal{K}(X) \;=\; \int_1^X \frac{\mathrm{d}X'}{X'}\, \frac{1}{n}\big(r + X'\big)^{\frac{1}{n}}\ , \tag{6.1}$$

where $X = (1 + \bar{z}z)^n |x|^2$ is an SU(n) invariant, and the z and x are the coordinates of the space. The resolution parameter r is defined, such that in the limit $r \rightarrow 0$ one retrieves the orbifold geometry. This space is a generalization of the Eguchi–Hanson space [114, 115]. From the Kähler potential all

Figure 6.1: The curvature (6.2) mimics a regularized delta–function.



geometrical quantities can be derived in the standard way, in particular, the curvature 2–form reads

$$\mathcal{R} = \frac{r}{r+X} \begin{pmatrix} e\,\bar{e} - \bar{e}\,e + \dfrac{1}{n}\,\dfrac{\bar{\epsilon}\,\epsilon}{r+X} & \dfrac{\bar{\epsilon}\,e}{\sqrt{r+X}} \\ \dfrac{\bar{e}\,\epsilon}{\sqrt{r+X}} & n\,\bar{e}\,e - \dfrac{n-1}{n}\,\dfrac{\bar{\epsilon}\,\epsilon}{r+X} \end{pmatrix} . \tag{6.2}$$

Here e and ϵ are the holomorphic vielbein 1–forms of $\mathbb{CP}^{n-1}$ and its complex line bundle. As this curvature is traceless, i.e. Ricci–flat, the blowup defines a non–compact Calabi–Yau space.

This geometry admits a U(1) gauge background satisfying the Hermitian Yang–Mills equations

$$i\mathcal{F}_V = \Big(\frac{r}{r+X}\Big)^{1-\frac{1}{n}} \Big(\bar{e}e - \frac{n-1}{n^2}\,\frac{1}{r+X}\,\bar{\epsilon}\epsilon\Big) H_V \ , \tag{6.3}$$

where $H_V = V^I H_I$ with H_I Cartan generators. The entries V^I are either all integers or half integers, and parameterize the embedding of the line bundles in the SO(32) (or $\mathrm{E}_8 \times \mathrm{E}_8$) gauge groups. Both the curvature (6.2) and the gauge field strength (6.3) become strongly peaked in the orbifold limit $r \to 0$ at $X = 0$; they mimic regularized delta functions, see figure 6.1.

Using the explicit geometry of the blowup of $\mathbb{C}^3/\mathbb{Z}_3$ with U(1) gauge bundle, we can construct string compactifications. Because both the geometry and its U(1) gauge background are given explicitly, the relevant integrals can be computed:

$$\int_{\mathbb{CP}^2} \frac{\operatorname{tr}\mathcal{R}^2}{(2\pi i)^2} = -n \int_{\mathbb{CP}^1 \ltimes \mathbb{C}} \frac{\operatorname{tr}\mathcal{R}^2}{(2\pi i)^2} = n(n+1) \ , \qquad \int_{\mathbb{CP}^p} \Big(\frac{i\mathcal{F}}{2\pi i}\Big)^p \ = \ -n \int_{\mathbb{CP}^{p-1}\ltimes\mathbb{C}} \Big(\frac{i\mathcal{F}}{2\pi i}\Big)^p \ = \ 1 \ . \tag{6.4}$$

The integrals over $\mathbb{CP}^p$ are taken at $X = 0$ integrating over p of the $n-1$ inhomogeneous coordinates of $\mathbb{CP}^{n-1}$. The integral over $\mathbb{CP}^{p-1} \ltimes \mathbb{C}$ corresponds to the integral over all values of $x \in \mathbb{C}$ and over $p-1$ inhomogeneous coordinates. The Bianchi identity integrated over $\mathbb{CP}^2$ has to vanish, using the above results this gives:

$$V^2 \ = \ 12 \ . \tag{6.5}$$

The same condition is found when integrating over $\mathbb{CP}^1 \ltimes \mathbb{C}$ and selects 7 allowed SO(32) blowup models listed in table 6.1. The spectra of these models can be compute using an index theorem. The

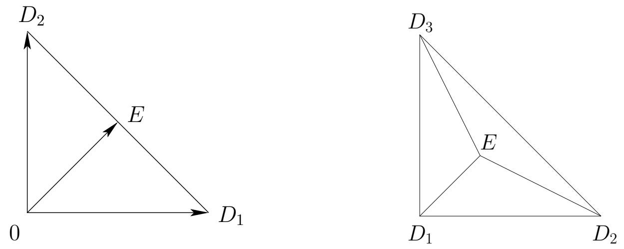
Table 6.1: The first column displays the heterotic $\mathbb{Z}_3$ SO(32) orbifold shifts. The U(1) bundles on the blowup are defined by the second column. The gauge groups of the heterotic orbifold models are listed in the next column. The one but last column contains the matter representations on the resolution. The last column gives the *additional* twisted matter on the orbifold.

Orbifold shift	Blowup shift	$G_{\text{orbifold}} = G_{\text{blow down}}$	Matter spectrum on the orbifold resolution	Additional twisted matter
$(0^{13},1^2,2)$	$(0^{12},1^3,3)$	SO(26) × U(3)	$\frac{1}{9}(\mathbf{26},\mathbf{3})+\frac{26}{9}(\mathbf{1},\overline{\mathbf{3}})+(\mathbf{26},\mathbf{1})$	$(\mathbf{1},\mathbf{1})$
	$(0^{13},2^3)$		$\frac{1}{9}(\mathbf{26},\overline{\mathbf{3}})+\frac{26}{9}(\mathbf{1},\mathbf{3})$	$(\mathbf{1},\mathbf{1})+(\mathbf{26},\mathbf{1})$
$(0^{10},1^4,2^2)$	$(0^{10},1^4,2^2)$	SO(20) × U(6)	$\frac{10}{9}(\mathbf{1},\overline{\mathbf{15}})+\frac{1}{9}(\mathbf{20},\mathbf{6})+3(\mathbf{1},\mathbf{1})$	
$(0^7,1^6,2^3)$	$(0^7,1^8,2)$	SO(14) × U(9)	$\frac{1}{9}(\mathbf{14},\mathbf{9})+\frac{1}{9}(\mathbf{1},\overline{\mathbf{36}})+(\mathbf{1},\overline{\mathbf{9}})$	
$(0^4,1^8,2^4)$	$(0^4,1^{12})$	SO(8) × U(12)	$\frac{1}{9}(\mathbf{8},\mathbf{12})+\frac{1}{9}(\mathbf{1},\overline{\mathbf{66}})$	$(\mathbf{1},\mathbf{1})+(\mathbf{8}_+,\mathbf{1})$
	$(\frac{1}{2}^{12},\frac{3}{2}^4)$		$\frac{1}{9}(\mathbf{8},\overline{\mathbf{12}})+\frac{1}{9}(\mathbf{1},\mathbf{66})+(\mathbf{8}_+,\mathbf{1})$	$(\mathbf{1},\mathbf{1})$
$(0^1,1^{10},2^5)$	$(\frac{1}{2}^{14},\frac{3}{2},\text{-}\frac{5}{2})$	SO(2) × U(15)	$\frac{11}{9}(\mathbf{15})+\frac{1}{9}(\overline{\mathbf{105}})+3(\mathbf{1})$	

multiplicities of the representations obtained from the branching of the adjoint of SO(32) (or $E_8 \times E_8$) via the multiplicity operator N_V. It takes the values: $N_V = \frac{1}{9},\ 1,\ \frac{26}{9} = 3 - \frac{1}{9}$. The multiplicity factor $\frac{1}{9} = \frac{3}{27}$ refers to untwisted (delocalized) states, while integral multiplicity factors correspond to states localized at the orbifold fixed point [64]. The table 6.1 compares the matter on the blowup with the heterotic orbifold spectrum in the blow down limit, and shows that only sometimes some vector–like matter is not recovered on the blowup. A similar analysis can also be performed for the blowup of $\mathbb{C}^2/\mathbb{Z}_2$, the results are consistent with the findings in [116].

One interesting application of the explicit construction of $\mathbb{C}^3/\mathbb{Z}_3$ orbifold blowup is to use it to explicitly verify the claim, that in blowup multiple anomalous U(1) are possible [100], even though it is known, that heterotic orbifold models always have at most a single anomalous U(1) , see e.g. [68]. The way out of this apparent paradox is, that a twisted state, with a non–vanishing VEV that induces the blowup, has to be reinterpreted as a model dependent axion. This model dependent axion cancel non–universal anomalies [117]. This result shed new light on the type–I/heteroric duality in four dimensions: The heterotic $\mathbb{C}^3/\mathbb{Z}_3$ orbifold model with gauge shift $V = (0^4, 1^8, \text{-}2^4)$ has gotten quite some attention in the past, because for this model there was a type–I dual $\mathbb{Z}_3$ orbifold model suggested [118–121]. However, because the Green–Schwarz anomaly cancellation is not the same in both models, and in particular, mediated by different fields, it had been questioned whether these models can really be dual [122]. Investigating this situation on the $\mathbb{C}^3/\mathbb{Z}_3$ blowup we developed here, we confirmed that the duality can realized in all fine print [117].

Figure 6.2: The left graph displays the toric diagram of $\text{Res}(\mathbb{C}^2/\mathbb{Z}_2)$. The right picture displays a projected view of the toric diagram of $\text{Res}(\mathbb{C}^3/\mathbb{Z}_3)$.



6.2 Toric resolutions

Next we turn the toric geometry approach to obtain resolutions of orbifold singularities. Here we do not have time to explain the toric geometry [123–125] in much detail. The rough idea of toric resolutions of orbifold singularities is to replace the orbifold action by invariances under $\mathbb{C}^*$ scalings of the coordinates z_i. To keep the dimensionality of the resolution equal to that of the orbifold, one needs to introduce as many extra coordinates x_p as complex scalings. Setting one of the homogeneous coordinates of the resolution to zero defines a codimension one hypersurface called a divisor. Ordinary divisors are defined by setting the original coordinates to zero $D_i = \{z_i = 0\}$, and exceptional divisors by setting the extra ones to zero $E_p = \{x_p = 0\}$. To each divisor we can associate a line bundle. As the first Chern class of a line bundle is a $(1,1)$–form, we reinterpret the divisors as $(1,1)$–forms themselves. Not all divisors are independent, because of so–called linear equivalence relations

$$\sum_i (v_i)_j \, D_i \;+\; \sum_p (w_p)_j \, E_p \;\sim\; 0 \,. \tag{6.6}$$

As there are as many such linear equivalence relations as ordinary divisors, we may take the exceptional divisors as a basis for the gauge background $\mathcal{F}_V$.

As hypersurfaces the divisors can intersect multiple times. These intersection numbers can be reinterpreted as integrals of the corresponding $(1,1)$–forms over the whole resolution. The intersections define the complete topology of the resolution. This topological information is conveniently summarized in the toric diagram: In a toric diagram the divisors are denoted as nodes, curves (intersection of two divisors) as lines between two nodes, and intersections of three different divisors as cones spanned by three nodes. Basic cones, the smallest possible cones, define intersections of three divisors with unit intersection number, while lines of three nodes correspond to intersection number zero. Together with the linear equivalence relations the toric diagram determines all (self–)intersections.

We have used these toric techniques to determine blowups of the orbifolds $\mathbb{C}^2/\mathbb{Z}_2$, $\mathbb{C}^2/\mathbb{Z}_3$, $\mathbb{C}^3/\mathbb{Z}_3$, $\mathbb{C}^3/\mathbb{Z}_4$ and $\mathbb{C}^3/\mathbb{Z}_2\times\mathbb{Z}_2'$ in [126]. Below, describe how to use toric geometry to heterotic blowup models for the orbifold $\mathbb{C}^3/\mathbb{Z}_3$ and $\mathbb{C}^3/\mathbb{Z}_4$ as interesting examples.

Toric resolution of $\mathbb{C}^3/\mathbb{Z}_3$

We illustrate the power of toric geometry by reproducing the results obtained using the explicit blowup of $\mathbb{C}^3/\mathbb{Z}_3$. The toric resolution of this orbifold has three ordinary divisors D_i, and a single exception one E. They satisfy the linear equivalence relations:

$$D_i \sim D_j \ , \qquad 3\,D_i + E \sim 0 \ , \tag{6.7}$$

From the toric diagram, right picture in figure 6.2, we infer the basic integrals and intersections: $D_1 D_2 E = D_2 D_3 E = D_3 D_1 E = 1$. The gauge field strength can be expanded as $\mathcal{F}_V = -\frac{1}{3}\,E\,H_V$. We obtained all the results of the explicit blowup. In particular, the Bianchi identity on the compact cycle E gives condition (6.5):

$$V^2 = \int_E \mathrm{tr}(i\mathcal{F}_V)^2 = \int_E \mathrm{tr}\,\mathcal{R}^2 = 12 \ . \tag{6.8}$$

The non–compact Bianchi identity follows immediately upon using the linear equivalence relation (6.7) and leads to the same condition. We have checked in [126] that we can also obtain all other results of the explicit resolution, including the full chiral spectrum.

Heterotic models on resolution of $\mathbb{C}^3/\mathbb{Z}_4$

The main advantage of toric geometry over explicit blowups lies in the fact that one can still use toric techniques in cases where no explicit blowup is known. To exemplify this we investigate the resolution of $\mathbb{C}^3/\mathbb{Z}_4$. In this case there are two exceptional divisors E_1 and E_2, which satisfy the linear equivalence relations

$$4\,D_1 + E_1 + 2\,E_2 \sim 0 \ , \qquad 4\,D_2 + E_1 + 2\,E_2 \sim 0 \ , \qquad 2\,D_3 + E_1 \sim 0 \ . \tag{6.9}$$

To define the integrals on the resolution of $\mathbb{C}^3/\mathbb{Z}_4$ we use the toric diagram, on the right hand side of figure 6.2, and obtain

$$D_1\,E_1\,E_2 = D_2\,E_1\,E_2 = D_1\,D_3\,E_1 = D_2\,D_3\,E_1 = 1 \ , \qquad D_1\,D_2\,E_2 = D_3\,E_1\,E_2 = 0 \ . \tag{6.10}$$

Via the linear equivalences this implies:

$$E_1^2\,E_2 = 0 \ , \ E_2^2\,E_1 = -2 \ , \ E_1^3 = 8 \ , \ E_2^3 = 2 \ . \tag{6.11}$$

The bottom edge of the toric diagram defines the toric diagram of the resolution of $\mathbb{C}^2/\mathbb{Z}_2$. The gauge background is expanded in terms of the exceptional divisors

$$\mathcal{F}_V = -\frac{1}{2}\,E_1\,H_1 - \frac{1}{4}\,(E_1 + 2\,E_2)H_2 \ , \tag{6.12}$$

where $H_1 = V_1^I H_I$, etc. In order to ensure that we can directly compute the spectrum on the non–compact resolution, we require that the Bianchi identities vanish on E_1, E_2 and the resolution of $\mathbb{C}^2/\mathbb{Z}_2$:

$$E_1: \ V_1^2 + V_1\cdot V_2 = 4 \ , \qquad E_2: \ V_1\cdot V_2 = -2 \ , \qquad \mathrm{Res}(\mathbb{C}^2/\mathbb{Z}_2): \ V_2^2 = 6 \ . \tag{6.13}$$

The matching between the heterotic orbifold models and the resolution models characterized by the shifts V_1 and V_2 is performed in table 6.2. All orbifold models, except number 4, are recovered in blowup. Model number 4 is not reproduce, because it does not have any first twisted sector, and therefore it can simply not be blown up. We have computed the complete spectrum and confirmed that all blowup models have anomaly free spectra mathiching with the orbifold models [126].

Table 6.2: This table compares the $\mathbb{C}^3/\mathbb{Z}_4$ orbifold gauge shift vector v, with the blowup vectors V_1 and V_2, that characterize the line bundle gauge background on the resolution.

orbifold shift $4\,v$	blowup vector V_2	blowup vector V_1	Nr.
$(0^{13},1^2,2)$	$(0^{13},1^2,2)$	$(0^{13},1^2,\text{-}2)$	1a
	$(0^{13},1^2,2)$	$(0^{12},2,\text{-}1^2,0)$	1b
	$(0^{13},1^2,2)$	$(0^{11},2,1,0^2,\text{-}1)$	1c
$(0^{11},1^2,2^3)$	$(0^{13},1^2,2)$	$(0^{10},1^4,\text{-}1^2)$	2a
	$(0^{13},1^2,2)$	$(0^{11},1^2,\text{-}2,0^2)$	2b
$(0^9,1^2,2^5)$	$(0^{13},1^2,2)$	$(0^8,1^5,0^2,\text{-}1)$	3a
	$(0^{13},1^2,2)$	$(0^9,1^4,\text{-}1^2,0)$	3b
$(0^7,1^2,2^7)$	–	–	4
$(0^{10},1^6)$	$(0^{10},1^6)$	$(0^{10},1^2,\text{-}1^4)$	5a
	$(0^{10},1^6)$	$(0^{13},1,\text{-}1,\text{-}2)$	5b
$(0^{10},1^5,3)$	$(0^{10},1^6)$	$(0^9,2,\text{-}1^2,0^4)$	6
$(0^8,1^6,2^2)$	$(0^{10},1^6)$	$(0^8,1^3,\text{-}1^3,0^2)$	7a
	$(0^{10},1^6)$	$(0^8,1^2,\text{-}2,0^5)$	7b
$(0^6,1^6,2^4)$	$(0^{10},1^6)$	$(0^6,1^4,\text{-}1^2,0^4)$	8

orbifold shift $4\,v$	blowup vector V_2	blowup vector V_1	Nr.
$(0^5,1^{10},2)$	$(0^{10},1^6)$	$\frac{1}{2}(\text{-}3,1^{10},\text{-}1^5)$	9
$(0^3,1^{10},2^3)$	$(0^{10},1^6)$	$\frac{1}{2}(1^{12},\text{-}1^3,\text{-}3)$	10
$(1^{14},2^2)$	$(0^{13},\text{-}2,1^2)$	$\frac{1}{2}(1^{15},\text{-}3)$	11
$(1^{13},\text{-}1,2^2)$	$(0^{13},1^2,2)$	$\frac{1}{2}(1^{15},\text{-}3)$	12a
	$(0^{13},1^2,2)$	$\text{-}\frac{1}{2}(\text{-}3,1^{15})$	12b
$\frac{1}{2}(1^3,3^{12},\text{-}3)$	$\frac{1}{2}(\text{-}3,1^{15})$	$\text{-}(0^{13},1^2,2)$	13a
	$\frac{1}{2}(1^{15},\text{-}3)$	$(0^{13},1^2,2)$	13b
	$\frac{1}{2}(1^{15},\text{-}3)$	$\frac{1}{2}(1^3,\text{-}1^{11},3,1)$	13c
$\frac{1}{2}(1^7,3^8,\text{-}3)$	$\frac{1}{2}(1^{15},\text{-}3)$	$(\text{-}1^5,1,0^{10})$	14a
	$\frac{1}{2}(1^{15},\text{-}3)$	$\frac{1}{2}(1^6,\text{-}1^8,\text{-}3,1)$	14b
	$\frac{1}{2}(1^{15},\text{-}3)$	$\frac{1}{2}(1^8,\text{-}1^7,3)$	14c
$\frac{1}{2}(1^{11},3^4,\text{-}3)$	$\frac{1}{2}(1^{15},\text{-}3)$	$(0^{10},1^3,\text{-}1^3)$	15
$\frac{1}{2}(1^{15},\text{-}3)$	$\frac{1}{2}(1^{15},\text{-}3)$	$(0^{13},\text{-}2,1^2)$	16a
	$\frac{1}{2}(1^{15},\text{-}3)$	$\frac{1}{2}(\text{-}1^{14},3,\text{-}1)$	16b

6.3 Conclusions

Even though in this work we have reviewed various methods to perform detailed computations on orbifolds, whether they in the end can truly be trusted remains somewhat questionable because orbifold curvature singularities make field theory analysis on them dangerous. Blowing up orbifold singularities provides a promising way to get around these problems. An important additional bonus for field theory is that on the such resolved singularities it is possible to compute the full localized spectrum. Field theory on orbifolds only catches the orbifold bulk states; one has to resort to string theory to predict the twisted states localized at the orbifold fixed points. Only for a very special class of orbifold singularities, $\mathbb{C}^n/\mathbb{Z}_n$, we have obtained blowups fully explicity. For more complicated singularities for which no explicit blowups are known, we showed that by resorting to toric geometry we are still able to recover the blowup models and determine their full chiral spectrum.

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