A Further Optimization of Crossover and Linear Barriers in Search Theory

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Introduction

Context

Since the start of World War II (WWII), technological advances have dramatically changed tactical and strategic operations. The science of Operations Research (OR) grew out of the need to solve problems related to evaluation and optimal use of these new technological advantages. Solving problems which occur in countering enemy technologies and newly implemented tactics was also important. An initial specialisation in OR was Search Theory. The tactics that were developed to search for the enemy played a very important role in the Allied efforts against German U-boats during WWII, see Fig. 1.

Figure 1. U-744 forced to surface, March the 6th, 1944 by depth charging in North Atlantic. Source: Naval Museum of Québec (http://www.mnq-nmq.org), with permission.

Interesting discussions regarding the search for submarines in the Bay of Biscay may be found in [2] and [3]. For example, it was discovered that aircraft maximised sighting distance by approaching at 45 degrees to U-boat tracks. Most search patterns ran either NW-SE or NE-SW across some assigned coverage area.

Nowadays, applications of Search Theory can be found practically everywhere. For example, the Navy and Air Force search for hostile submarines, Special Forces search for terrorist groups, Unmanned Aerial Vehicles search for nuclear plants or launching facilities of opposing forces. Of course, there are also many non-military applications, such as the search and rescue of drowning persons and counter drug operations.

Using mathematical and probabilistic models, Search Theory has developed several interesting search patterns that optimise the probability of detecting a target. For an overview of classical search theory, we refer to [1].

Many papers nowadays focus on multi-agent systems and simulations. For applications, we refer to [2] and [3]. In this contribution we follow an analytical approach.

The reason for doing so, is that we are able to demonstrate analytically that further optimization is possible by slightly modifying the well-known *crossover barrier* (Fig. 2) and *linear barrier* (Fig. 3), where the search area is a *lane* (i.e. a Southwards going channel), see [4].

We assume that targets intend to traverse this lane Southwards. We also assume that target speed U is constant along its path. This assumption is not far from reality because, after reaching cruise level, the target usually maintains a steady speed. A further assumption is that we know the intent and capabilities of the target. More precisely, we assume that we know its speed. Its position, however, is unknown, so arbitrary. Examples are rescuing a person floating in the water, detecting fast drug boats, etc.

We assume the observer is protecting the lane while moving at speed V through the lane according to some fixed pattern. Any target that closes the observer to within his *sweep radius* R is detected. So the observer's detecting device is binary: the target is either detected or not detected. A further assumption is that there is enough time for the observer to detect targets.

The *crossover barrier* starts on the left of the lane and crosses to the opposite side (track OA) in such a way that its Southwards movement equals that of a hypothetical target which simultaneously moves from B to A. Next the observer moves Northwards (track AB), crosses the lane to the left (track BC), and finally moves Northwards to the starting-point (track CO). In this way a *butterfly* search pattern is created. After completing one basic movement the pattern is repeated several times. The *crossover* model is chosen, when the speed of the observer is greater than that of the target.

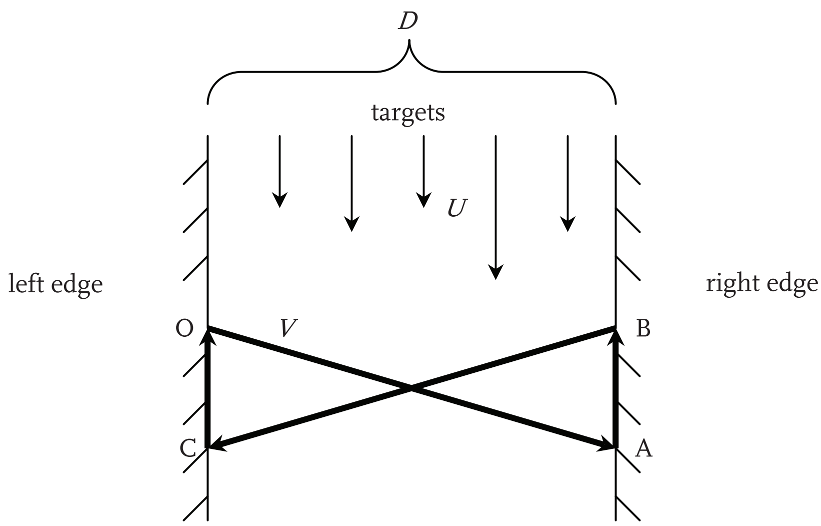

Figure 2. Crossover barrier (changing course at the edge)

The *linear barrier* moves in a straight line from West to East, i.e. its course is perpendicular to that of the targets. If the observer reaches the edge of the lane, it will reverse course. In this way a *linear* search pattern is created. After completing one basic movement the pattern is repeated several times. The *linear* model is chosen, when the speed of the observer is less than or equal to that of the target.

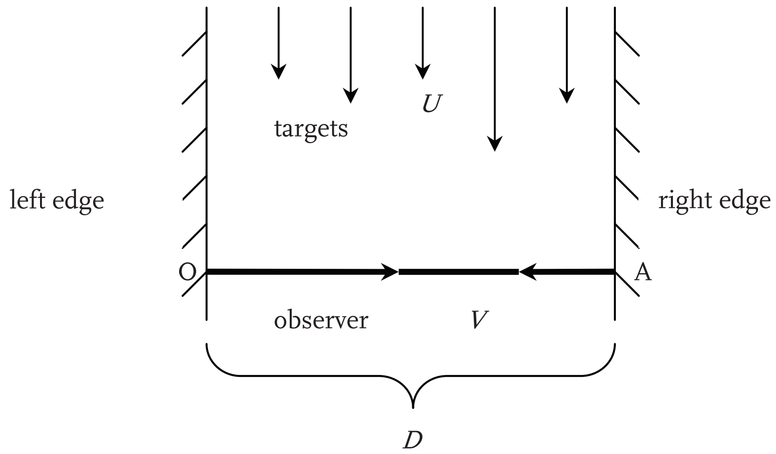

Figure 3. Linear barrier (reversing course at the edge)

Problem definition
It is well-known that instead of changing course exactly at the edge, changing course when the sweep radius reaches the edge (i.e. changing course at distance R from the edge) generally yields higher probabilities of detection.

In this contribution we shall investigate whether or not an even higher probability of detection may be obtained, by turning at distance xR from the edge. The *turning-factor* x is some real number between 0 and 1. So, if $x = 0$, the *barrier* changes course at the edge. If $x = 1$, the *barrier* changes course when the sweep radius reaches the edge.

We will discuss two questions:

1. If the barrier changes course at distance R from the edge, does this situation – compared with changing course at the edge – always lead to a higher probability of detection?
2. If the barrier changes course at the edge or at distance R from the edge, does one of these two situations lead to a maximum probability of detection?

The construction of this contribution will be as follows: first in two different sections, we present the *crossover barrier* as well as the *linear barrier*. In both models we will go into three scenarios:

- the barrier changes course at the edge;
- the barrier changes course when the sweep radius reaches the edge;
- the barrier changes course at distance xR from the edge.

In the section *Results and Discussion*, an overview of the results will be presented for both models. A discussion on the choice between the two models is also carried on. Finally in the section *Conclusions*, the answers to the questions as formulated above will be summarized.

We will calculate all probabilities of detection choosing a *lane width* D with magnitude 24 nautical miles (NM), while *sweep radius* R equals 4 NM. Hence a comparison between the two models and the three scenarios can be made. We will vary the ratio of observer's speed V and target's speed U between just greater than one and four (*crossover* model), respectively half and four (*linear* model). This quotient V/U is called the *speed ratio* and is denoted by ρ. Hence $1 < \rho \leq 4$ (*crossover* model), respectively $0.5 \leq \rho \leq 4$ (*linear* model).

Crossover barrier patrol

Changing course at the edge

We assume $V > U$ (i.e. $\rho > 1$). If $V < U$ or $V \approx U$, the *crossover* model is not practicable, because the angle – in relation to the horizontal axis – chosen by the *crossover barrier* is not defined or close to $\frac{1}{2}\pi$. The latter is not desirable because it will cost the *barrier* too much time to reach the opposite side of the lane. The lane is D wide, while the sweep radius of the observer equals R. We assume $D > 2R$, because if $D \leq 2R$ the observer could restrict himself to a position in the middle of the lane.

Let t_1 be the time it takes for the observer to reach the opposite side of the lane. We can use Pythagoras' Theorem (applied in a right-angled triangle with hypotenuse Vt_1 and catheti D and Ut_1) to determine $t_1 = \dfrac{D}{\sqrt{V^2 - U^2}}$.

So, observer and target move according to $\begin{pmatrix} D \\ -Ut_1 \end{pmatrix}$, respectively $\begin{pmatrix} 0 \\ -Ut_1 \end{pmatrix}$ from their initial position.

To determine the probability of detection, we keep the position of the target fixed with only the observer and its detection circle moving *relatively* to the target. The relative movement of the first leg (track OA) is obtained by calculating the difference of the observer's absolute movement and that of the target:

$$\begin{pmatrix} D \\ -Ut_1 \end{pmatrix} - \begin{pmatrix} 0 \\ -Ut_1 \end{pmatrix} = \begin{pmatrix} D \\ 0 \end{pmatrix}.$$

Let t_2 be the time it takes for the observer to proceed up the channel. The length of this *upsweep* is equal to Ut_1. Hence:

$$Vt_2 = Ut_1 = \frac{D}{\sqrt{\left(\frac{V}{U}\right)^2 - 1}}.$$

So, it follows:

$$t_2 = \frac{D}{V\sqrt{\rho^2 - 1}}.$$

Hence the relative movement of the second leg (track AB) satisfies:

$$\begin{pmatrix} 0 \\ Vt_2 \end{pmatrix} - \begin{pmatrix} 0 \\ -Ut_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{D}{\sqrt{\rho^2-1}} + \frac{D}{\rho\sqrt{\rho^2-1}} \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{D}{\rho}\sqrt{\frac{\rho+1}{\rho-1}} \end{pmatrix}.$$

The expression:

$$S = \frac{D}{\rho}\sqrt{\frac{\rho+1}{\rho-1}} \tag{1}$$

is called the *spacing* and is denoted by S.
Hence the relative movement of the observer in relation to the target is given by $\begin{pmatrix} D \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ S \end{pmatrix}, \begin{pmatrix} -D \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ S \end{pmatrix}$. In this way a *meander* search pattern is created, see Fig. 4, where a strip with width $2R$ around the relative track will be swept.

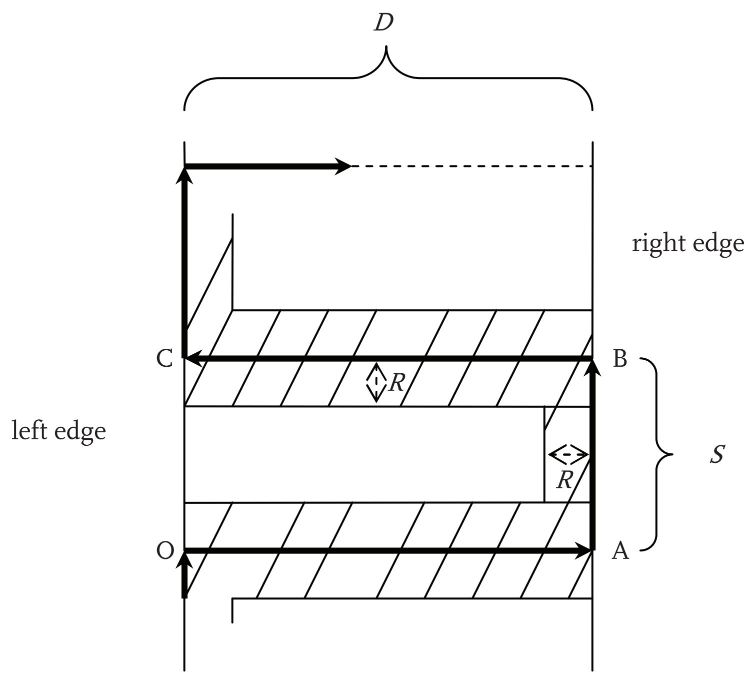

Figure 4. The relative area swept for the crossover barrier changing course at the edge ($R \leq \frac{1}{2}S$)

Since the target has a fixed and random position, the probability of detection is determined by taking the ratio of the shaded area and the total area. We can – in view of the regularity in the pattern of the movement – restrict ourselves to one distinctive part of the relative track, i.e. the track OA → AB → BC in the rectangle OABC. The dimensions of this rectangle are *lane width* D and *spacing* S.

If $R \geq \frac{1}{2}S$, the total area will be swept. So the probability of detection P_{det} will be equal to 1.
If $R \leq \frac{1}{2}S$, then:

$$P_{\text{det}} = \frac{2DR + (S-2R)R}{DS}. \tag{2}$$

Changing course when the sweep radius reaches the edge
If the observer changes course when the sweep radius reaches the edge, the geometry will be more complex, but the general idea is the same. In a distinctive part of the relative movement the ratio of the swept area and the total area will be calculated. As Fig. 5 shows, the swept area consists of rectangles and sectors of a circle.

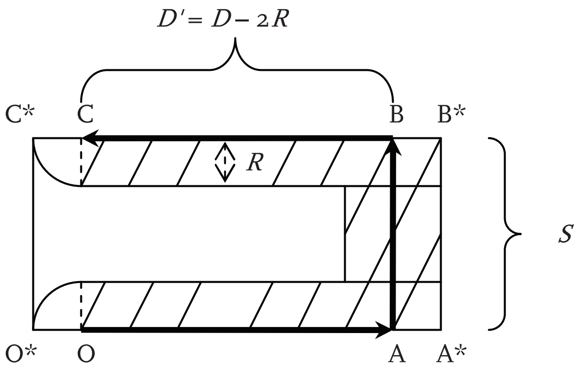

Figure 5. The relative area swept for the crossover barrier changing course when the sweep radius reaches the edge $(R \leq S/2)$

The area of the two sectors of a circle equals $\frac{1}{2}\pi R^2$. There are also two rectangles with dimensions $D - R$ and R and a rectangle with dimensions $S - 2R$ and $2R$.

Because OA equals $D - 2R$, *spacing* S is calculated on the basis of $D' = D - 2R$. Hence:

$$S = \frac{D'}{\rho}\sqrt{\frac{\rho + 1}{\rho - 1}}\,. \tag{3}$$

If $R \leq \frac{1}{2}S$, then:

$$P_{\text{det}} = \frac{2R(D - R) + 2R(S - 2R) + \frac{1}{2}\pi R^2}{DS}\,. \tag{4}$$

If $R \geq \frac{1}{2}S$, the rectangles overlap, as do the sectors, see Fig. 6.

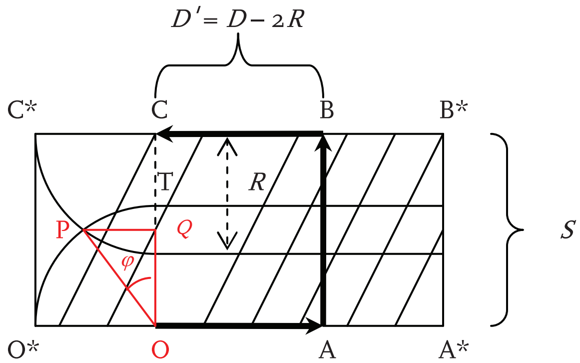

Figure 6. The relative area swept for the crossover barrier changing course when the sweep radius reaches the edge $(R \geq \frac{1}{2}S)$

The area of the overlap of the sectors of the circle is obtained by first calculating sector OPT, then by calculating triangle OPQ. The overlap is twice the difference of these two results, i.e. the difference of twice sector OPT and twice triangle OPQ.

Using $\varphi = \arccos(\frac{S}{2R})$ we obtain the following probability of detection:

$$P_{\text{det}} = \begin{cases} \dfrac{2R(D-R)+2R(S-2R)+\frac{1}{2}\pi R^2}{DS} & \text{, if } R \leq \frac{S}{2} \\ \dfrac{(D-R)S+\frac{1}{2}\pi R^2 - R^2 \arccos(\frac{S}{2R}) + \frac{S}{2}\sqrt{R^2-\frac{S^2}{4}}}{DS} & \text{, if } R \geq \frac{S}{2} \end{cases} \tag{5}$$

Changing course at distance xR from the edge

If the observer changes course at distance xR ($0 \leq x \leq 1$) from the edge, geometry gets even more complex, but the general idea is the same. The ratio of the swept area and the total area will be calculated in a distinctive part of the relative movement.

Again we distinguish cases $R \leq ½S$ and $R \geq ½S$.

If $R \leq ½S$ the swept part consists of rectangles and truncated sectors of a circle, see Fig. 7.

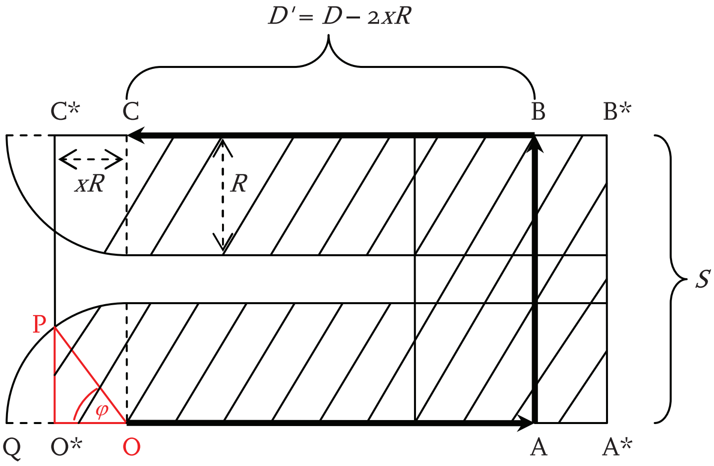

Figure 7. The relative area swept for the crossover barrier changing course at distance xR from the edge ($R \leq ½S$)

If $R \geq ½S$, the rectangles overlap, as do the truncated sectors, see Fig. 8. If PQ is longer than xR, the overlap is so much that the swept area is equal to OABC. Hence the detection probability equals 1. This occurs when $xR <$ PQ $= \sqrt{R^2 - \frac{S^2}{4}}$, i.e. $x < \sqrt{1-(\frac{S}{2R})^2}$.

If $x \geq \sqrt{1-(\frac{S}{2R})^2}$, we refer to Fig. 8.

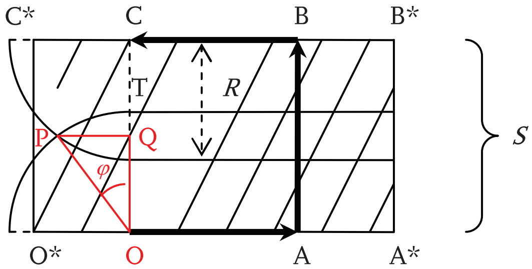

Figure 8. The relative area swept for the crossover barrier changing course at distance xR from the edge ($R \geq ½S$)

Calculations similar to (5) lead to the following probability of detection:

$$P_{\det} = \begin{cases} \dfrac{2(D - xR)R + (S - 2R)(R + xR) + \frac{1}{2}\pi R^2 - R^2 \arccos x + R^2 x\sqrt{1 - x^2}}{DS}, & \text{if } R \leq \frac{S}{2} \\[2ex] \dfrac{(D - xR)S + R^2\left(\frac{1}{2}\pi - \arccos x + x\sqrt{1-x^2} - \arccos\left(\frac{S}{2R}\right)\right) + \frac{S}{2}\sqrt{R^2 - \frac{S^2}{4}}}{DS}, & \text{if } x \geq \sqrt{1 - \left(\frac{S}{2R}\right)^2} \end{cases} \tag{6}$$

Substituting x = 0, respectively x = 1 in (6) it is easy to check that the probability of detection satisfies (2), respectively (4) and (5).

Linear barrier patrol

Reversing course at the edge

As mentioned before: a *linear barrier* is preferred if $V < U$ or $V \approx U$. The *linear barrier* moves from West to East, respectively from East to West. Targets always move Southwards. If t is the time it takes for the observer to reach the opposite side of the lane, then the relative movement of the track satisfies:

$$\begin{pmatrix} Vt \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ -Ut \end{pmatrix} = \begin{pmatrix} Vt \\ Ut \end{pmatrix}, \text{ respectively } \begin{pmatrix} -Vt \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ -Ut \end{pmatrix} = \begin{pmatrix} -Vt \\ Ut \end{pmatrix}.$$

In this way a *ladder* search pattern is created as a relative track. A strip with width $2R$ around this track will be swept.

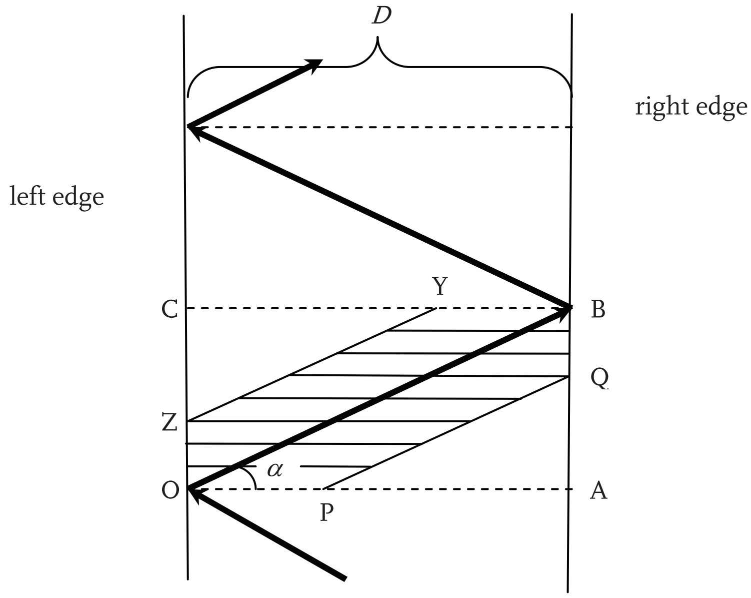

Figure 9. The relative swept area for the linear barrier changing course at the edge (R/sin $\alpha \leq D$)

Since the target has a fixed and random position, the probability of detection is determined by taking the ratio of the shaded area and the total area. We can – in view of the regularity in the pattern – restrict ourselves to one distinctive part of the relative track, i.e. the track OB in rectangle OABC. See Fig. 9[1].

[1] We assume P to be to the left of A, i.e. OA > OP. Hence $D > \frac{R}{\sin\alpha} = R\sqrt{1+\rho^2}$. If $D \leq R\sqrt{1+\rho^2}$, the swept area coincides with OABC. So, the probability of detection equals 1. If D = 24 and R = 4, this will only happen if $\rho \geq 6.0$. Hence we can disregard this situation.

Using $\sin\alpha = \dfrac{U}{\sqrt{U^2+V^2}} = \dfrac{1}{\sqrt{1+\rho^2}}$ we obtain:

$$P_{\text{det}} = \frac{2R}{D}\sqrt{1+\rho^2} - \frac{R^2}{D^2}\left(1+\rho^2\right). \tag{7}$$

Reversing course when the sweep radius reaches the edge

If the observer reverses course when the sweep radius reaches the edge, the geometry will be more complex, but the general idea is the same. In a distinctive part of the relative movement the ratio of the swept area and the total area will be calculated. As Fig. 10 shows, the swept area consists of a hexagon and two sectors of a circle[1].

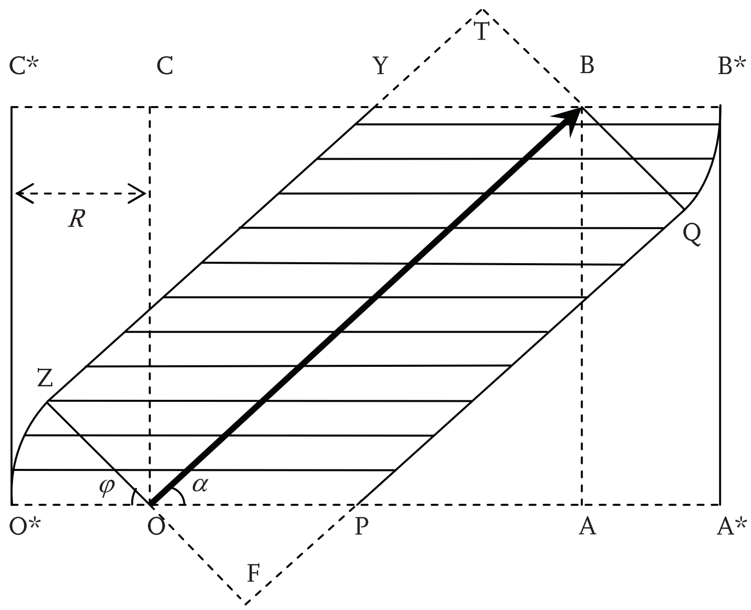

Figure 10. The relative area swept for the linear barrier reversing course when the sweep radius reaches the edge

Calculating the swept area (i.e. hexagon OPQBYZ and sectors O*OZ and B*BQ) and the whole area (i.e. rectangle O*A*B*C*) and using $\alpha = \operatorname{arccot}\rho$ gives:

$$P_{\text{det}} = \frac{2R}{D}\sqrt{1+\rho^2} + \frac{R^2}{D(D-2R)}\left\{\left(\tfrac{1}{2}\pi - \operatorname{arccot}\rho\right)\rho - \rho^2\right\}. \tag{8}$$

Reversing course at distance xR from the edge

If the observer reverses course at distance xR ($0 \le x \le 1$) from the edge, the geometry gets even more complex, but the general idea is the same.

Two cases arise:

<u>Case A</u>: Z inside the lane (see Fig. 11)

[1] We assume Q to be above AA*. If $\rho \le 4$, $D = 24$ and $R = 4$ (so AB ≥ 4), this assumption will be satisfied.

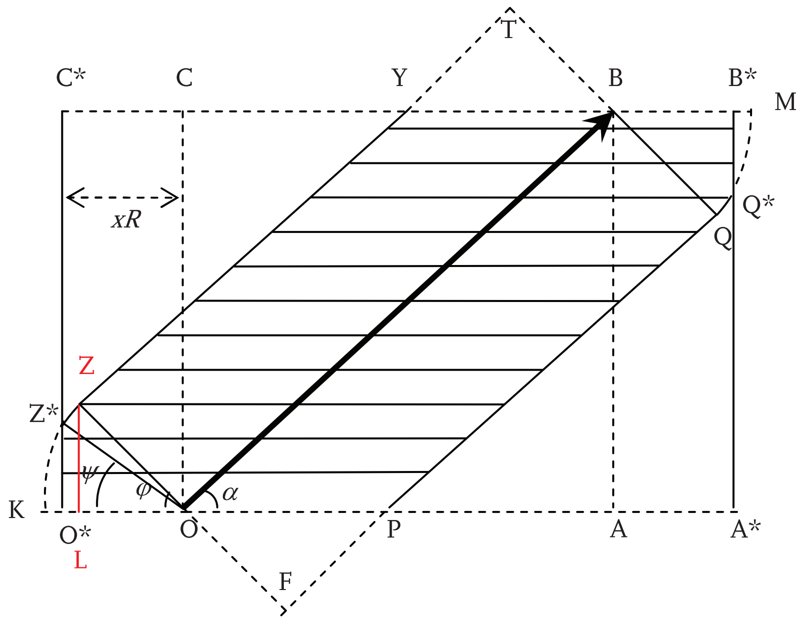

Figure 11. The swept part for the linear barrier reversing course at distance xR from the edge ($x \geq \frac{1}{\sqrt{1+\rho^2}}$)

This occurs when $O^*O = xR \geq LO = R\cos\varphi = R\cos(\tfrac{1}{2}\pi - \alpha) = R\sin\alpha = \frac{R}{\sqrt{1+\rho^2}}$, i.e. $x \geq \frac{1}{\sqrt{1+\rho^2}}$. In this case, the swept area consists of hexagon OPQBYZ and truncated sectors OO*Z*Z and BB*Q*Q.

<u>Case B</u>: Z outside the lane (see Fig. 12)

This occurs when $x < \frac{1}{\sqrt{1+\rho^2}}$.

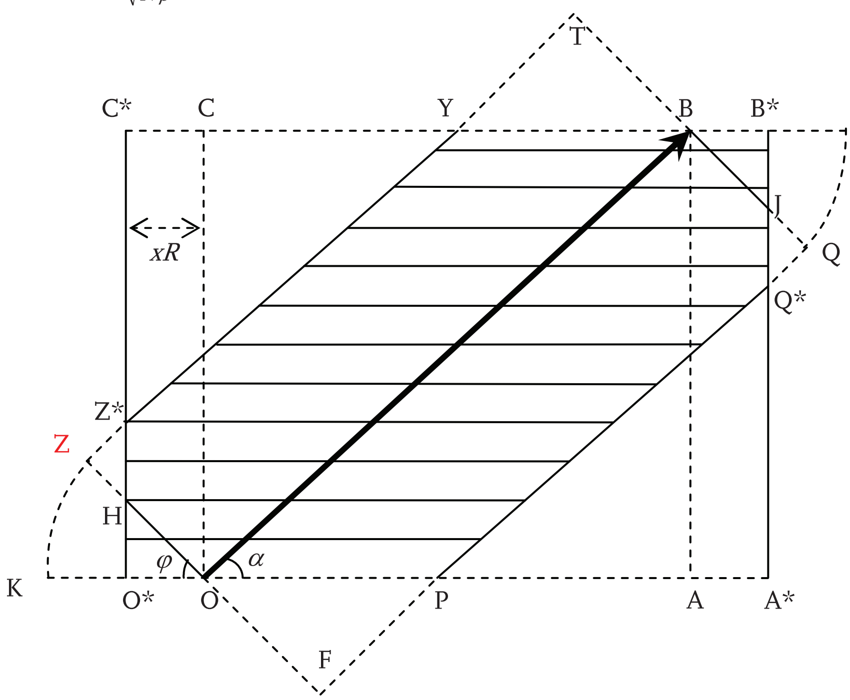

Figure 12. The swept part for the linear barrier reversing course at distance xR from the edge ($x < \frac{1}{\sqrt{1+\rho^2}}$)

In this case, the swept area is obtained by first calculating rectangle ZFQT, then subtracting triangles FOP, BTY, HZZ* and JQQ*, and then adding triangles HOO* and BB*J.

We can prove that the probability of detection satisfies:

$$P_{\det} = \begin{cases} \dfrac{2R}{D}\sqrt{1+\rho^2} + R^2 \left\{ \dfrac{\left(\frac{1}{2}\pi - \operatorname{arccot}\rho - \arccos x + x\sqrt{1-x^2} \right)\rho - \rho^2}{D(D-2xR)} \right\} & \text{, if } x \geq \frac{1}{\sqrt{1+\rho^2}} \\ \dfrac{2R}{D}\sqrt{1+\rho^2} + \dfrac{2xR^2\sqrt{1+\rho^2} - R^2\rho^2 - (x^2+1)R^2}{D(D-2xR)} & \text{, if } x < \frac{1}{\sqrt{1+\rho^2}} \end{cases} \tag{9}$$

Results and discussion

We have chosen $D = 24$, $R = 4$ and $1 < \rho \leq 4$ (*crossover*), respectively $0.5 \leq \rho \leq 4$ (*linear*). The probability of detection is calculated for several values of the *speed ratio* ρ and the *turning-factor x*.

Crossover barrier patrol
In Table 1, we also mention the magnitude of the *spacing* S_0 (corresponding to $x = 0$), respectively S_1 (corresponding to $x = 1$).

Table 1. Probability of detection as a function of speed ratio ρ and turning-factor x for the crossover barrier

	ρ	1.5	2.0	2.5	3.0	3.5	4.0
x	S_0	35.8	20.8	14.7	11.3	9.2	7.7
0.00		0.353	0.487	0.621	0.756	0.891	1.000
0.05		0.363	0.498	0.633	0.768	0.905	1.000
0.10		0.372	0.508	0.644	0.781	0.918	1.000
0.15		0.382	0.519	0.656	0.793	0.931	1.000
0.20		0.392	0.530	0.667	0.806	0.945	1.000
0.25		0.401	0.541	0.679	0.819	0.959	1.000
0.30		0.411	0.551	0.691	0.832	0.973	1.000
0.35		0.421	0.562	0.703	0.844	0.987	1.000
0.40		0.431	0.573	0.715	0.857	0.998	1.000
0.45		0.440	0.584	0.727	0.870	0.999	1.000
0.50		0.450	0.595	0.739	0.883	0.998	1.000
0.55		0.460	0.606	0.751	0.896	0.998	1.000
0.60		0.470	0.616	0.762	0.909	0.998	1.000
0.65		0.480	0.627	0.774	0.922	0.997	1.000
0.70		0.489	0.638	0.786	0.935	0.997	1.000
0.75		0.499	0.649	0.797	0.947	0.996	1.000
0.80		0.509	0.659	0.809	0.959	0.994	0.999
0.85		0.518	0.669	0.820	0.971	0.993	0.998
0.90		0.527	0.679	0.831	0.977	0.990	0.996
0.95		0.536	0.689	0.841	0.974	0.987	0.993
1.00		0.545	0.698	0.850	0.970	0.982	0.988
	S_1	23.9	13.9	9.8	7.5	6.1	5.2

As *speed ratio* ρ increases, *spacing* S_x decreases. The swept area of the rectangle will assume growing importance in relation to the total area, see Figs. 4 – 8. Hence the probability of detection will increase as *speed ratio* ρ increases. This result is true for all values of $x \in [0.00; 1.00]$.

Comparing changing course at the edge and changing course when the sweep radius reaches the edge leads to the conclusion that – at a given ρ – *spacing* S will be smaller. Changing course at distance R from the edge thus leads to a higher probability of detection. $\rho = 4$ is an exception: when changing course at the edge *spacing* S will be less than $2R$. So the probability of detection equals 1. Hence the probability of detection – in the case of changing course at distance R from the edge – will turn out to be smaller[1]. If $\rho \leq 2.5$, the probability of detection is maximal, if the observer changes course at distance R from the edge. If $\rho = 3$ or $\rho = 3.5$, then changing course at distance R from the edge gives a higher probability of detection when compared with changing course at the edge, but the probability of detection is maximal at a turning-distance smaller than R. As ρ increases, the optimal turning-distance decreases. If $\rho = 4$, the probability of detection is maximal, if the observer changes course at the edge[2].

Linear barrier patrol

As *speed ratio* ρ increases, the probability of detection increases too (see Table 2). This is illustrated in Fig. 9: as ρ increases, gradient α decreases, as does AB. The swept area will assume growing importance in relation to the total area. This holds for every $x \in [0.00; 1.00]$.

Table 2. Probability of detection as a function of speed ratio ρ and turning-factor x for the linear barrier

x \ ρ	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
0.00	0.338	0.416	0.511	0.606	0.696	0.776	0.845	0.902
0.05	0.340	0.419	0.514	0.610	0.700	0.780	0.849	0.906
0.10	0.343	0.422	0.518	0.614	0.704	0.785	0.853	0.909
0.15	0.345	0.425	0.521	0.618	0.709	0.789	0.857	0.913
0.20	0.348	0.428	0.524	0.622	0.713	0.793	0.861	0.916
0.25	0.350	0.430	0.528	0.626	0.717	0.797	0.865	0.920
0.30	0.352	0.433	0.531	0.630	0.721	0.801	0.869	0.923
0.35	0.354	0.436	0.535	0.633	0.725	0.805	0.873	0.927
0.40	0.356	0.438	0.538	0.637	0.729	0.810	0.877	0.930
0.45	0.358	0.441	0.541	0.641	0.733	0.814	0.881	0.933
0.50	0.360	0.444	0.544	0.645	0.737	0.818	0.884	0.936
0.55	0.362	0.446	0.548	0.649	0.741	0.821	0.888	0.939
0.60	0.363	0.448	0.551	0.652	0.745	0.825	0.891	0.941
0.65	0.365	0.451	0.554	0.656	0.749	0.829	0.894	0.943
0.70	0.366	0.453	0.557	0.659	0.752	0.832	0.896	0.944
0.75	0.368	0.455	0.560	0.662	0.755	0.834	0.898	0.945
0.80	0.369	0.457	0.562	0.665	0.758	0.837	0.899	0.944
0.85	0.370	0.459	0.565	0.668	0.760	0.838	0.899	0.943
0.90	0.371	0.461	0.567	0.670	0.762	0.839	0.899	0.941
0.95	0.372	0.462	0.568	0.671	0.762	0.838	0.896	0.936
1.00	0.372	0.462	0.569	0.671	0.761	0.835	0.891	0.929

Comparing both situations – reversing course at the edge and reversing course at distance R from the edge – leads to the conclusion that – at a given ρ – the latter has a higher probability of detection. If $\rho \leq 1.50$, the probability of detection is maximal, if the observer

[1] If $\rho = 4$, then – if changing course at the edge – S equals 7.7, i.e. less than $2R$. So, the probability of detection equals 1. *Spacing* S is a function of D' (see Eq. (3)): as D' increases, S increases too. Hence a probability of detection with magnitude 1 will be obtained at a higher value of ρ. Therefore the mentioned exception is only valid when $D = 24$ and $R = 4$. If $D = 20$, the exception is true if ρ is greater than, or equal to 3.5.

[2] If $\rho = 4$ the maximum probability of detection equals 1. The mentioned value of x is only one possible solution. Every choice of $x \in [0.00; 0.65]$ leads to a maximum detection probability.

changes course at distance R from the edge. If $\rho \geq 2.0$, then changing course at distance R from the edge gives a higher probability of detection when compared with changing course at the edge, but the probability of detection is maximal at a turning-distance smaller than R. As ρ increases, the optimal turning-distance decreases.

Choosing between crossover and linear

As mentioned before: the choice between the *crossover* model and the *linear* model depends on the magnitude of the observer's speed in relation to that of the target. If $V < U$, a *linear* model has to be chosen. If $V \approx U$, a *linear* model is preferred, since – in case of a *crossover barrier* – the observer's absolute course would be almost Southwards: if $V = 10.5$ and $U = 10$, then observer's (absolute) course equals 162 degrees. The latter is not desirable, because it will cost the *barrier* too much time to reach the other side of the lane. If $V >> U$, a *crossover* model is preferred. Crossing will be almost West/Eastwards: if $V = 30$ and $U = 10$, then observer's (absolute) course equals 110 degrees.

The decision between *crossover* and *linear* is not only influenced by *speed ratio*, but also by $D' = D - 2xR$. In this contribution – we have chosen $D = 24$ and $R = 4$ – only the *turning-factor* x varies ($0 \leq x \leq 1$). Hence $16 \leq D' \leq 24$.

Linking both models in such a way that the model with maximum detection probability is chosen, we can give a clear idea of the dependence on *speed ratio* ρ and *turning-factor* x (see Table 3). Detection probabilities using the *linear* model are represented in green, those using the *crossover* model in blue. The maximum detection probability is represented in red.

Table 3. Probability of detection as a function of speed ratio ρ and turning-factor x for the linear barrier, respectively the crossover barrier

x \ ρ	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
0.00	0.338	0.416	0.511	0.606	0.696	0.776	0.891	1.000
0.05	0.340	0.419	0.514	0.610	0.700	0.780	0.905	1.000
0.10	0.343	0.422	0.518	0.614	0.704	0.785	0.918	1.000
0.15	0.345	0.425	0.521	0.618	0.709	0.793	0.931	1.000
0.20	0.348	0.428	0.524	0.622	0.713	0.806	0.945	1.000
0.25	0.350	0.430	0.528	0.626	0.717	0.819	0.959	1.000
0.30	0.352	0.433	0.531	0.630	0.721	0.832	0.973	1.000
0.35	0.354	0.436	0.535	0.633	0.725	0.844	0.987	1.000
0.40	0.356	0.438	0.538	0.637	0.729	0.857	0.998	1.000
0.45	0.358	0.441	0.541	0.641	0.733	0.870	0.999	1.000
0.50	0.360	0.444	0.544	0.645	0.739	0.883	0.998	1.000
0.55	0.362	0.446	0.548	0.649	0.751	0.896	0.998	1.000
0.60	0.363	0.448	0.551	0.652	0.762	0.909	0.998	1.000
0.65	0.365	0.451	0.554	0.656	0.774	0.922	0.997	1.000
0.70	0.366	0.453	0.557	0.659	0.786	0.935	0.997	1.000
0.75	0.368	0.455	0.560	0.662	0.797	0.947	0.996	1.000
0.80	0.369	0.457	0.562	0.665	0.809	0.959	0.994	0.999
0.85	0.370	0.459	0.565	0.669	0.820	0.971	0.993	0.998
0.90	0.371	0.461	0.567	0.679	0.831	0.977	0.990	0.996
0.95	0.372	0.462	0.568	0.689	0.841	0.974	0.987	0.993
1.00	0.372	0.462	0.569	0.698	0.850	0.970	0.982	0.988

If $\rho \leq 1.5$, the *linear* model is preferred, if $\rho \geq 3.5$ the *crossover* model is preferred. If $2.0 \leq \rho \leq 3.00$, the situation is mixed: if the *barrier* is changing course near the edge

($x \approx 0$), the *linear* model is preferred; if the *barrier* is changing course near distance R from the edge ($x \approx 1$), the *crossover* model is preferred.

Conclusions hold in this particular situation (D = 24, R = 4), but it is possible to demonstrate that similar conclusions can be drawn for other values of *lane width* and *sweep radius*.

Conclusions

In the *Introduction* to this contribution two questions were formulated:

1. If the *barrier* changes course at distance R from the edge, does this situation – compared with changing course at the edge – always lead to a higher probability of detection?
2. If the *barrier* changes course at the edge or at distance R from the edge, does one of these two situations lead to a maximum probability of detection?

We have investigated two models: the *crossover* model and the *linear* model. In both models we made a distinction between changing course at the edge, changing course when the sweep radius reaches the edge and changing course at an alternating distance.

If $D = 24$ and $R = 4$, the following conclusions can be drawn:

1. If $\rho \leq 3.5$, changing course when sweep radius reaches the edge will – compared with changing course at the edge – lead to a higher probability of detection. If $\rho \geq 4.0$, changing course at the edge will give a better result.
2. If $\rho \leq 2.5$, changing course when sweep radius reaches the edge will lead to a maximum probability of detection. If $\rho \geq 4.0$, changing course at the edge will give the best result. If $\rho = 3.0$ or $\rho = 3.5$, maximum probability of detection will be obtained when the *barrier* changes course at distance from the edge less than R. As *speed ratio* increases, turning-distance from the edge will decrease.

Following an analytical approach, we were able to demonstrate that – obtaining the probability of detection – further optimization is possible. More precisely, the probability of detection increases by a few percent under certain circumstances, if we choose the turning-distance of the *barrier* carefully.

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